Date: 4/8/25

Chp: Chp. 4:5 → Differentials

Obj:
- understand Differential
- Find differential, estimate change, absolute change, relative error, & % error

*Derivative = Change in a function
= denoted as $\frac{dy}{dx}$

*Differential = Change in a variable
= the dy or dx

*Must have a differentiable function in order to have differentials.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$\Delta x \cdot f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x} \cdot \Delta x$$

$$f'(x)\cdot \Delta x \approx f(x+\Delta x)-f(x)$$

$\Delta x = dx$

$$f'(x)\cdot dx = \underbrace{f(x+\Delta x)-f(x)}_{\Delta y = dy}$$

$$\rightarrow f'(x)\cdot dx = dy$$

Differentials

Let $y=f(x)$ be a differentiable function. The differential dx is the independent variable. The differential dy is the dependent variable & can be found by

$$f'(x)\cdot dx = dy$$

- Every differentiation formula has a differential formula:

Ex: $f(x)=\sin x \quad f'(x)=\cos x \cdot x'$

$$\frac{d}{dx}\sin u = \cos u \cdot u' \rightarrow d(\sin u) = \cos u \cdot u' = \cos u \cdot du$$

$$\frac{d}{dx}(u \pm v) = \frac{d}{dx}u \pm \frac{d}{dx}v$$

$$\rightarrow \frac{du}{dx} \pm \frac{dv}{dx} \text{ or } d(u \pm v) \text{ or } du \pm dv$$

Ex. 1 – Find the differential.

a) $y = t^3 - 4t^2 + 7t$

$dy = 3t^2 \cdot dt - 8t \cdot dt + 7 \cdot dt$

$3t^2 dt - 8t dt + 7 dt$

$dt(3t^2 - 8t + 7)$

b) $w = x^2 \sin(2x)$

$dw = 2x \cdot dx \cdot \sin(2x) + \cos(2x)(2) \cdot dx \cdot x^2$

$dw = 2x\,dx \sin 2x + 2x^2\,dx \cos 2x$

$dw = dx(2x \sin 2x + 2x^2 \cos 2x)$

c) $f(z) = e^{3-z^4}$

d) ~~$d(\tan 2x) =$~~

$d(x) = \tan 2x$

e) $d(x) = \dfrac{x}{x+1}$

Ex. 2 – Find the differential dy & evaluate it for the given x & dx.

a) $y = \sin 3x$, $x = \pi$, $dx = -0.02$

$dy = \cos 3x \cdot 3 \cdot dx$

$dy = \cos 3(\pi) \cdot 3 \cdot -0.02$

$dy = -1(3)(-0.02)$

$dy = 0.06$

b) $y = x^5 + 37x$, $x = 1$, $dx = 0.01$

c) $x + y = xy$, $x = 2$, $dx = 0.05$

$2 + y = 2y$

$2 = y$

$x = xy - y$

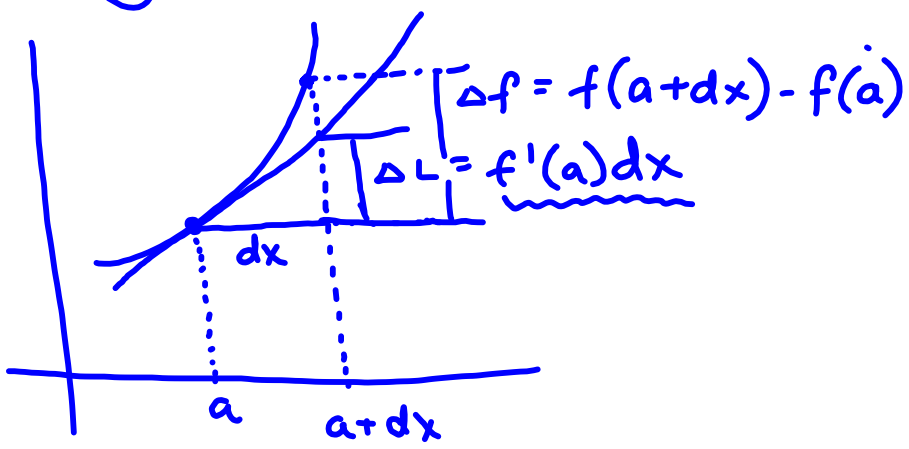
$x = y(x-1)$

$\frac{x}{x-1} = y$

$x\,dx - dx - x\,dx$

$\frac{1\,dx(x-1) - 1\,dx(x)}{(x-1)^2}$

Estimating Change



Differential Estimate of Change

Let $f(x)$ be differentiable @ $x=a$.
The approximate change in f when x changes from a to $a+dx$ is

$$df = f'(a)dx$$

Absolute Change/Error

$$\underbrace{f(x+dx)}_{\text{(new)}} - \underbrace{f(x)}_{\text{(orig)}} = dy$$

Relative Change/Error

Ratio of $\frac{dV}{V}$

Percent Change/Error

Relative Change · 100

Ex. 3 - The radius of a circle increases from 10 m to 10.1 m. Use dA to estimate the increase in the circle's area A. Compare the estimate to the true change in A (ΔA). Find absolute error/propagated error.

$A = \pi r^2$

$dA = 2\pi r \cdot dr$

$dA = 2\pi(10)(0.1)$

$dA = 2\pi \ m^2$

$\pi(10.1)^2 - \pi(10)^2$

$102.01\pi - 100\pi$

$2.01\pi \ m^2$

error $= 0.01\pi \ m^2$

Ex. 4 – The radius of a ball bearing is measured to be 0.7 inches. If the measurement is correct to within 0.01 inches, estimate the error in the volume of the ball bearing.

$V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2 dr$

$dV = 4\pi(0.7)^2(0.01)$

$dV = 0.0196\pi \text{ in}^3$

$\Delta V = \frac{4}{3}\pi(0.71)^3 - \frac{4}{3}\pi(0.7)^3$

$\Delta V = 0.477\pi - 0.457\pi$

$\Delta V = 0.02\pi \text{ in}^3$

$\text{error} = 0.004\pi \text{ in}^3$ (0.0428)

$\text{rel. change} = \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = \frac{3\,dr}{r} = \frac{3(0.01)}{0.7}$

% change error = 4.28%

b) Find relative error.

c) Find the % error.

Ex 5 – A scientist discovered a formula to predict how much the radius of a partially clogged artery has to be expanded to get normal blood flow back.

$$V = kr^4$$

Volume of fluid flowing through a small tube in a unit of time at a fixed pressure (k) times the tube's radius (r)^4.

How will a 10% increase in r affect V?

$$V = kr^4$$

$$dV = 4kr^3\,dr$$

$$\frac{4kr^3\,dr}{kr^4}$$

$$\frac{4\,dr}{r} = \frac{4(0.1)}{r}$$

40%

Ex. 4 – Use differentials to find relative maximum error.

a) area of a circle with radius of 10 ± 0.01 inch.

$$\frac{dA}{A} = \frac{2\pi r \cdot dr}{\pi r^2} = \frac{2dr}{r} \qquad A = \pi r^2, \; dA = 2\pi r\, dr$$

$$\frac{2(\pm 0.01)}{10} = \pm 0.002 \rightarrow \pm 0.2\%$$

b) Vol of a sphere w/ radius of 8 ± 0.3 inches!

$$\frac{dV}{V} =$$

c) Vol of a cube w/ side lengths of 15 ± 0.2 cm

Homework:

p. 243 (#31-39 odds, 44, 46)