Date: 4/3/24

Chp: Chp. 4:4 → Modeling & Optimization

Obj: Be able to find greatest profit, least cost, least time, largest volume, etc (Differential Calculus)

Strategy

1. Read & understand the problem
2. Make a sketch
3. Find a formula for the quantity to be maximized or minimized.
4. Express the quantity in Step 3 as a function in one variable.
5. State the restrictions (Domain)
6. Find any critical pts and endpts.

Ex. 1 - Find 2 #s whose sum is 20 & product is as large as possible

$f(x) = \overset{10}{x}(20 - \overset{10}{x})$

$f(x) = 20x - x^2$

$f'(x) = 20 - 2x$

$0 = 20 - 2x$

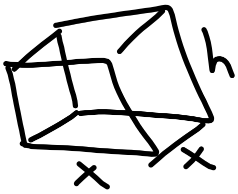
$-20 = -2x$

$10 = x$

$10(10) = 100$

Ex. 3 - Manufacture an open box w/ a square base and a $SA = 108\ in^2$. What dimensions will produce max volume?

$V \geq 0$
$0 \leq x \leq \sqrt{108}$



$V = x^2 h$

$x^2 \leq 108$
$x \leq \sqrt{108}$

$V = \frac{x^2}{1}\left(\frac{108 - x^2}{4x}\right)$

$SA = 4xh + x^2$

$108 = 4xh + x^2$

$108 - x^2 = 4xh$

$\frac{108 - x^2}{4x} = h$

$V = \frac{108x^2 - x^4}{4x}$

$V' = \frac{(216x - 4x^3)(4x) - 4(108x^2 - x^4)}{(4x)^2}$

$V' = \frac{(216x - 4x^3)(4x) - 432x^2 + 4x^4}{16x^2}$

$V' = \frac{864x^2 - 16x^4 - 432x^2 + 4x^4}{16x^2}$

$V' = \frac{432x^2 - 12x^4}{16x^2} = \frac{12x^2(36 - x^2)}{16x^2}$

$V' = \frac{3(36 - x^2)}{4}$

$0 = \frac{3(36 - x^2)}{4}$

$0 = 3(36 - x^2)$

$0 = 36 - x^2$

$-36 = -x^2$

$36 = x^2$

$\pm 6 = x$

$6 = x$

$v(0) = 0$ or $\cancel{0}$

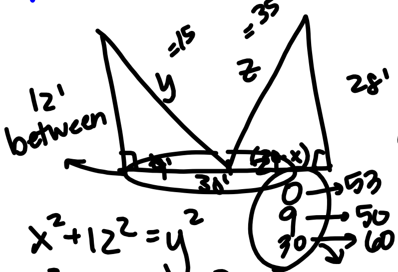
$v(6) = 108$

$v(\sqrt{108}) = 0$

$x = 6$

$6 \times 6 \times 3$ in

Ex. 4 - 2 posts (12 ft. tall & 28 ft tall) stand 30 ft apart. They are to be stayed by 2 wires attached to a single stake running from ground level to the top of each post. Where should the stake be placed to use the least wire?



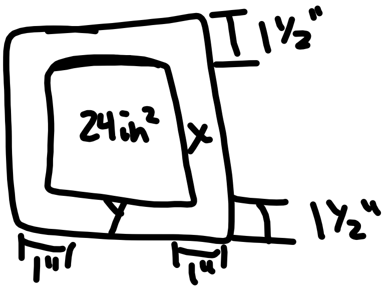
$$(30-x)^2+28^2=z^2$$
$$900-60x+x^2+784=z^2$$
$$1684-60x+x^2=z^2$$
$$\sqrt{1684-60x+x^2}=z$$

$$x^2+12^2=y^2$$
$$x^2+144=y^2$$
$$\sqrt{x^2+144}=y$$

$$w=\sqrt{x^2+144}+\sqrt{1684-60x+x^2}$$
$$w=(x^2+144)^{1/2}+(1684-60x+x^2)^{1/2}$$

$$w'=\frac{1}{2}(x^2+144)^{-1/2}(2x)+\frac{1}{2}(1684-60x+x^2)^{-1/2}(-60+2x)$$
$$w'=x(x^2+144)^{-1/2}+(-30+x)(1684-60x+x^2)^{-1/2}$$
$$w'=\frac{x}{\sqrt{x^2+144}}+\frac{-30+x}{\sqrt{1684-60x+x^2}}$$
$$0=\frac{x}{\sqrt{x^2+144}}+\frac{-30+x}{\sqrt{1684-60x+x^2}}$$
$$-\frac{-30+x}{\sqrt{1684-60x+x^2}}=\frac{x}{\sqrt{x^2+144}}$$
$$\left[(30-x)\sqrt{x^2+144}\right]^2=\left[x\sqrt{1684-60x+x^2}\right]^2$$
$$(30-x)^2(x^2+144)=x^2(1684-60x+x^2)$$
$$(900-60x+x^2)(x^2+144)=1684x^2-60x^3+x^4$$
$$900x^2-60x^3+x^4+129600-8640x+144x^2=1684x^2-60x^3+x^4$$
$$1044x^2\cancel{-60x^3}+\cancel{x^4}+129600-8640x=1684x^2\cancel{-60x^3}+\cancel{x^4}$$
$$0=640x^2+8640x-129600$$
$$0=320(2x^2+27x-405)$$
$$0=2x^2+27x-405$$
$$x=9,\ \cancel{-22.5}$$

Ex. 5 – A rectangular page is to have 24 in² of print. Margins at top & bottom are 1½ in and sides are 1 in. What are the dimensions ~~so that the least amount~~ of the paper? ~~is used?~~



$$A = (\overset{6}{x}+3)(\overset{4}{y}+2)$$

$$24 = \overset{6}{x}y$$

$$\frac{24}{x} = y$$

$$\boxed{9" \times 6"}$$

$$A = (x+3)\left(\frac{24}{x}+2\right)$$

0, 6, 24

$$A = 24 + 2x + \frac{72}{x} + 6$$

$$A = 30 + 2x + \frac{72}{x} \rightarrow 72x^{-1}$$

$$A' = 2 - 72x^{-2}$$

$$A' = 2 - \frac{72}{x^2}$$

$$0 = 2 - \frac{72}{x^2}$$

$$-2 = -\frac{72}{x^2}$$

$$-2x^2 = -72$$

$$x^2 = 36$$

$$x = 6$$

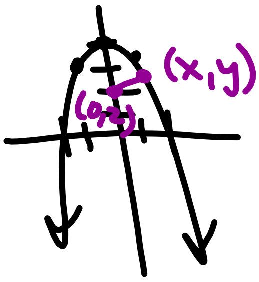
$$A(0) = \cancel{0}$$

$$A(6) = 54 \; *$$

$$A(24) = 81$$

Ex. 6 – Which pts on the graph of $y=4-x^2$ are closest to $(0,2)$?

$\sqrt{\frac{3}{2}} \approx 1.2$



$$D=\sqrt{(x-x)^2+(y-y)^2}$$

$$D=\sqrt{(x-0)^2+(y-2)^2}$$

$$D=\sqrt{(x-0)^2+(4-x^2-2)^2}$$

$$D=\sqrt{x^2+(2-x^2)^2}$$

$$D=\sqrt{x^2+4-4x^2+x^4}$$

$$D=\sqrt{x^4-3x^2+4}=(x^4-3x^2+4)^{1/2}$$

$$D'=\frac{1}{2}(x^4-3x^2+4)^{-1/2}(4x^3-6x)$$

$$D'=\frac{2x^3-3x}{\sqrt{x^4-3x^2+4}}$$

$$0=\frac{2x^3-3x}{\sqrt{x^4-3x^2+4}}$$

$$0=2x^3-3x$$

$$0=x(2x^2-3)$$

$$0=2x^2-3$$

$$3=2x^2$$

$$\frac{3}{2}=x^2$$

$$\pm\sqrt{\tfrac{3}{2}}=x$$

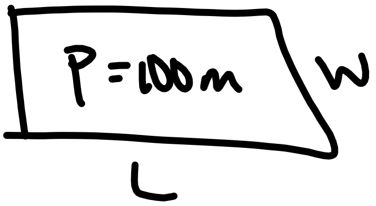
$D(0)=2$

$D\left(\sqrt{\tfrac{3}{2}}\right)=1.3228$

$D\left(-\sqrt{\tfrac{3}{2}}\right)=1.3228$

$\left(\sqrt{\tfrac{3}{2}}, 1.3\right)$ $\left(-\sqrt{\tfrac{3}{2}}, 1.3\right)$

Ex. 7 – Find the dimensions of a rect. w/ a perimeter of 100 m whose area is as large as possible.



$A = L(W)$

$A = (50-W)(W)$

$A = 50W - W^2$

$A' = 50 - 2W$

$0 = 50 - 2W$

$-50 = -2W$

$25_m = W$

$25_m \times 25_m$

$100 = 2L + 2W$

$100 - 2W = 2L$

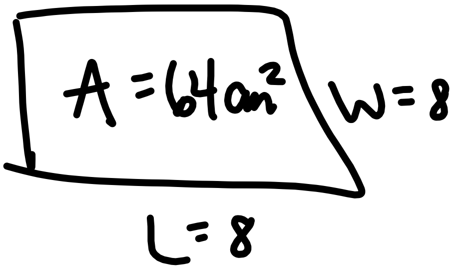
$50 - \underset{25}{W} = L$

$A(0) = 0$

$A(25) = 625$

$A(100) = -5000$

Ex.8 – What is the smallest perimeter possible for a rectangle whose area is 64 cm^2 ?



$$64 = L(w)$$

$$_{8}\frac{64}{w} = L$$

$$P = 2L + 2w$$

$$P = 2\left(\frac{64}{w}\right) + 2w$$

$$P = \frac{128}{w} + 2w$$

$$P' = \frac{\cancel{0(w)} - 1(128)}{w^2} + 2$$

$$P' = \frac{-128}{w^2} + 2$$

$$P' = \frac{-128 + 2w^2}{w^2}$$

$$0 = \frac{-128 + 2w^2}{w^2}$$

$$0 = -128 + 2w^2$$

$$128 = 2w^2$$

$$\sqrt{64} = \sqrt{w^2}$$

$$8 = w$$

$$P(0) = \varnothing$$

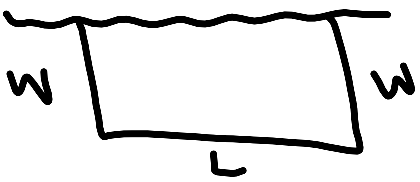
$$P(8) = 32$$

$$P(64) = 130$$

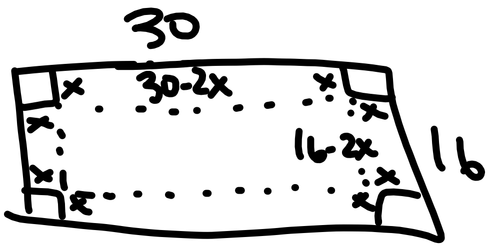
$$P = 32\text{ cm}$$

Ex. 9- A farmer has 200 m of fencing & wants to fence off a rectangular field that borders a straight river. What are the dimensions of the field that has the largest area?

(no fence along river)



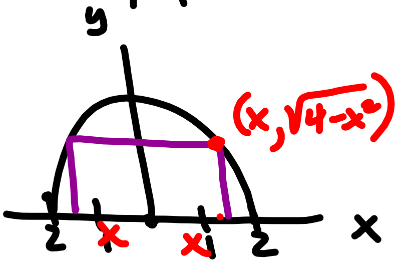
Ex. 10 - An open box is to be made from a 16 cm by 30 cm piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest possible volume?



$V = l \cdot w \cdot h$

Ex. 11 - Find the least amount of lumber that will be needed to form an open box with a square base and a capacity of 32 m^3.

Ex. 12 – Find the area of the largest rectangle that can be inscribed in a semicircle of radius 2.



$A = l \cdot w$

$l = 2x = 2\sqrt{2}$ $(\sqrt{2})$

$w = \sqrt{4-x^2} = \sqrt{2}$ $(\sqrt{2})$

$A = 2x\sqrt{4-x^2}$

$(x-h)^2 + (y-k)^2 = r^2$ (with $h = 0$, $k = 0$)

$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \sqrt{4-x^2}$

$A = 2x(4-x^2)^{1/2}$

$A' = 2(4-x^2)^{1/2} + \frac{1}{2}(4-x^2)^{-1/2}(-2x)(2x)$

$A' = 2(4-x^2)^{1/2} - \frac{2x^2}{(4-x^2)^{1/2}}$

$0 = 2(4-x^2)^{1/2} - \frac{2x^2}{(4-x^2)^{1/2}}$

$\frac{2x^2}{(4-x^2)^{1/2}} = 2(4-x^2)^{1/2}$

$2x^2 = 2(4-x^2)$

$x^2 = 4 - x^2$

$2x^2 = 4$

$x^2 = 2$

$x = \sqrt{2}$

$A = 2\sqrt{2}(\sqrt{2})$

$A = 2(2)$

$A = 4 \text{ units}^2$

Mason = #1, #4, #6

Luke = #2, #3, #5

Nathan = #7, #8, #3

Ex. 13 – A rectangle has its base on the x-axis and its upper two vertices on the parabola $y=16-x^2$. What is the largest area the rectangle can have, and what are its dimensions?