Properties of Exponential and Logarithmic Equations

Let \( a \) be a positive real number such that \( a \neq 1 \), and let \( x \) and \( y \) be real numbers. Then the following properties are true:

1. \( a^x = a^y \) if and only if \( x = y \)
2. \( \log_a x = \log_a y \) if and only if \( x = y \) (\( x > 0, \ y > 0 \))

Inverse Properties of Exponents and Logarithms

<table>
<thead>
<tr>
<th>Base ( a )</th>
<th>Natural Base ( e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_a (a^x) = x )</td>
<td>( \ln(e^x) = x )</td>
</tr>
<tr>
<td>( a^{\log_a x} = x )</td>
<td>( e^{(\ln x)} = x )</td>
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Solving Exponential and Logarithmic Equations

1. To solve an exponential equation, first isolate the exponential expression, then take the logarithm of both sides of the equation and solve for the variable.
2. To solve a logarithmic equation, first isolate the logarithmic expression, then exponentiate both sides of the equation and solve for the variable.

For Instance: If you wish to solve the equation, \( \ln x = 2 \), you exponentiate both sides of the equation to solve it as follows:

\[
\begin{align*}
\ln x &= 2 & \text{Original equation} \\
\exp(\ln x) &= e^2 & \text{Exponentiate both sides} \\
x &= e^2 & \text{Inverse property}
\end{align*}
\]

Or you can simply rewrite the logarithmic equation in exponential form to solve (i.e. \( \ln x = 2 \) if and only if \( e^2 = x \)).

Note: You should always check your solution in the original equation.

Example 1:

Solve each equation.

a. \( 4^{x+2} = 64 \)

Solution:

\[
\begin{align*}
4^{x+2} &= 64 & \text{Original Equation} \\
4^{x+2} &= 4^3 & \text{Rewrite with like bases} \\
x + 2 &= 3 & \text{Property of exponential equations} \\
x &= 1 & \text{Subtract 2 from both sides}
\end{align*}
\]

The solution is 1. Check this in the original equation.

b. \( \ln(2x - 3) = \ln 11 \)

Solution:

\[
\begin{align*}
\ln(2x - 3) &= \ln 11 & \text{Original Equation} \\
2x - 3 &= 11 & \text{Property of logarithmic equations} \\
2x &= 14 & \text{Add 3 to both sides} \\
x &= 7 & \text{Divide both sides by 2}
\end{align*}
\]

The solution is 7. Check this in the original equation.

Example 2:

Solve \( 5 + e^{x+1} = 20 \).

Solution:

\[
\begin{align*}
5 + e^{x+1} &= 20 & \text{Original Equation} \\
e^{x+1} &= 15 & \text{Subtract 5 from both sides} \\
\ln e^{x+1} &= \ln 15 & \text{Take the logarithm of both sides} \\
x + 1 &= \ln 15 & \text{Inverse Property} \\
x &= -1 + \ln 15 \approx 1.708 & \text{Subtract 1 from both sides}
\end{align*}
\]

Check:

\[
\begin{align*}
5 + e^{x+1} &= 20 & \text{Original Equation} \\
e^{x+1} &\approx 14.999 & \text{Substitute 1.708 for } x \\
5 + e^{1.708+1} &\approx 20 & \text{Substitute 1.708 for } x \\
5 + e^{2.708} &\approx 20 & \text{Simplify} \\
5 + 14.999 &\approx 20 & \text{Solution checks}
\end{align*}
\]
Example 3:
Solve the exponential equations.

a. \(2^x = 7\)

b. \(4^{x-3} = 9\)

c. \(2e^x = 10\)

Solutions:

**Method 1:**

a. \(2^x = 7\)  
   Original Equation
   \[
   \log 2^x = \log 7 \quad \text{Take the logarithm of both sides}
   \]
   \[
   x \log 2 = \log 7 \quad \text{Property of Logarithms}
   \]
   \[
   x = \frac{\log 7}{\log 2} \approx 2.807 \quad \text{Solve for } x
   \]

b. \(4^{x-3} = 9\)  
   Original Equation
   \[
   \log 4^{x-3} = \log 9 \quad \text{Take the logarithm of both sides}
   \]
   \[
   (x - 3) \log 4 = \log 9 \quad \text{Property of Logarithms}
   \]
   \[
   x - 3 = \frac{\log 9}{\log 4} \quad \text{Divide both sides by } \log 4
   \]
   \[
   x = 3 + \frac{\log 9}{\log 4} \approx 4.585 \quad \text{Solve for } x
   \]

c. \(2e^x = 10\)  
   Original Equation
   \[
   e^x = 5 \quad \text{Divide both sides by 2}
   \]
   \[
   \ln e^x = \ln 5 \quad \text{Take the logarithm of both sides}
   \]
   \[
   x = \ln 5 \approx 1.609 \quad \text{Inverse Property}
   \]

**Example 4:**
Solve \(2 \log_4 x = 5\).

Solution:

\[
2 \log_4 x = 5 \quad \text{Original Equation}
\]
\[
\log_4 x = \frac{5}{2} \quad \text{Divide both sides by 2}
\]
\[
4^{5/2} = x \quad \text{Change to exponential form}
\]
\[
x = 32 \quad \text{Simplify}
\]

**Example 6:**
Solve \(20 \ln 0.2x = 30\).

Solution:

\[
20 \ln 0.2x = 30 \quad \text{Original Equation}
\]
\[
\ln 0.2x = 1.5 \quad \text{Divide both sides by 20}
\]
\[
0.2x = e^{1.5} \quad \text{Change to exponential form}
\]
\[
x = 5e^{1.5} \approx 22.408 \quad \text{Divide both sides by 0.2}
\]

**Example 5:**
Solve \(3 \log x = 6\).

Solution:

\[
3 \log x = 6 \quad \text{Original Equation}
\]
\[
\log x = 2 \quad \text{Divide both sides by 3}
\]
\[
10^2 = x \quad \text{Change to exponential form}
\]
\[
x = 100 \quad \text{Simplify}
\]

**Example 7:** Solving a Logarithmic Equation using Exponentiation

Solve \(\log_3 2x - \log_3 (x - 3) = 1\)

Solution:

\[
\log_3 2x - \log_3 (x - 3) = 1 \quad \text{Original Equation}
\]
\[
\log_3 \frac{2x}{x-3} = 1 \quad \text{Condense the left side}
\]
\[
3^{\log_3 \frac{2x}{x-3}} = 3^1 \quad \text{Exponentiate both sides}
\]
\[
\frac{2x}{x-3} = 3 \quad \text{Inverse Property}
\]
\[
2x = 3x - 9 \quad \text{Multiply both sides by } x - 3
\]
\[
x = 9 \quad \text{Solve for } x
\]
Practice Problems

Solve the following equations:
Remember that the arguments of all logarithms must be greater than 0. Also exponentials in the form of $a^x$ will be greater than 0. Be sure to check all your answers in the original equation.

1. $3^{x-1} = 81$
2. $8^x = 4$
3. $e^x = 5$
4. $-14 + 3e^x = 11$
5. $-6 + \ln 3x = 0$
6. $\log(3x + 1) = 2$
7. $\ln x - \ln 3 = 4$
8. $2 \ln 3x = 4$
9. $5^{x+2} = 4$
10. $\ln(x + 2)^2 = 6$
11. $4^{-3x} = 0.25$
12. $2e^{2x} - 5e^x - 3 = 0$
13. $\log_7 3 + \log_7 x = \log_7 32$
14. $2\log_6 4x = 0$
15. $\log_2 x + \log_2(x - 3) = 2$
16. $\log_2(x + 5) - \log_2(x - 2) = 3$
17. $4\ln(2x + 3) = 11$
18. $\log x - \log 6 = 2\log 4$
19. $2^x = 64$
20. $5^x = 25$
21. $4^{x-3} = \frac{1}{16}$
22. $3^{x-2} = 81$
23. $\log_3 x = 5$
24. $\log_4 x = 3$
25. $\log_2 2x = \log_2 100$
26. $\ln(x + 4) = \ln 7$
27. $\log_3(2x + 1) = 2$
28. $\log_5(x - 10) = 2$
29. $3^x = 500$
30. $8^x = 1000$
31. $\ln x = 7.25$
32. $\ln x = -0.5$
33. $2e^{0.5x} = 45$
34. $100e^{-0.6x} = 20$
35. $12(1 - 4^x) = 18$
36. $25(1 - e^t) = 12$
37. $\log 2x = 1.5$
38. $\log_2 2x = -0.65$
39. $\frac{1}{3}\log_2 x + 5 = 7$
40. $4\log_5(x + 1) = 4.8$
41. $\log_2 x + \log_2 3 = 3$
42. $2\log_4 x - \log_4(x - 1) = 1$
Practice Problems Answers

1. 5
2. \( \frac{2}{3} \)
3. 1.609
4. 2.120
5. 134.476
6. 33
7. 163.794
8. 2.463
9. -1.139
10. 18.086, -22.086
11. \( \frac{1}{3} \)
12. 1.099
13. \( \frac{32}{3} \)
14. \( \frac{1}{4} \)
15. 4
16. 3
17. 6.321
18. 96
19. 6
20. 2
21. 1
22. 6
23. 243
24. 64
25. 50
26. 3
27. 4
28. 35
29. 5.66
30. 3.32
31. 1408.10
32. 0.61
33. 6.23
34. 2.68
35. No Solution
36. -0.65
37. 15.81
38. 0.32
39. 64
40. 5.90
41. \( \frac{8}{3} \)
42. 2