Graphing More Complicated Rational Functions

Learning Objective

Students will graph more complicated rational functions, including those with point discontinuities and slant asymptotes, and use more complicated rational functions to model problem situations.

Math Processes and Practices

MPP8 Generalizing

Language Objective

Students describe and explain the key features of the graph of a complicated rational function.

Online Resources

An extra example for each Explain section is available online.

Engage

Essential Question: What features of the graph of a rational function should you identify in order to sketch the graph? How do you identify those features?

Possible answer: Identify vertical asymptotes and "holes," based on the factors of the function's denominator, horizontal asymptotes, and slant asymptotes, based on an analysis of the function's end behavior; *x*-intercepts based on the factors of the function's numerator, where the graph lies above or below the *x*-axis, based on an analysis of the signs of the factors in both the numerator and denominator.

Preview: Lesson Performance Task

View the Engage section online. Discuss the photo and how the radius of a baseball is a function of its volume. Then preview the Lesson Performance Task.

8.2 Graphing More Complicated Rational Functions

Resource Locker

Essential Question: What features of the graph of a rational function should you identify in order to sketch the graph? How do you identify those features?

Explore 1 Investigating Domains and Vertical Asympotes of More Complicated Rational Functions

You know that the rational function $f(x) = \frac{1}{x-2} + 3$ has the domain $\left\{x \mid x \neq 2\right\}$ because the function is undefined at $x = 2$. Its graph has the vertical asymptote $x = 2$ because as $x \to 2^+$ (x approaches 2 from the right), $f(x) \to +\infty$, and as $x \to 2^-$ (x approaches 2 from the left), $f(x) \to -\infty$. In this Explore, you will investigate the domains and vertical asymptotes of other rational functions.

(A) Complete the table by identifying each function's domain based on the *x*-values for which the function is undefined. Write the domain using an inequality, set notation, and interval notation. Then state the equations of what you think the vertical asymptotes of the function's graph are.

Function	Domain	Possible Vertical Asymptotes
$f(x) = \frac{x+3}{x-1}$	$x < 1$ or $x > 1$ $\left\{x \mid x \neq 1\right\}$ $(-\infty, 1) \cup (1, +\infty)$?	? $x = 1$
$f(x) = \frac{(x+5)(x-1)}{x+1}$	$x < -1$ or $x > -1$ $\left\{x \mid x \neq -1\right\}$ $(-\infty, -1) \cup (-1, +\infty)$?	? $x = -1$
$f(x) = \frac{x-4}{(x+1)(x-1)}$	$x < -1$ or $-1 < x < 1$ or $x > 1$ $\left\{x \mid x \neq -1 \text{ and } x \neq 1\right\}$ $(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$?	? $x = -1, x = 1$
$f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$	$x < -2$ or $-2 < x < 3$ or $x > 3$ $\left\{x \mid x \neq -2 \text{ and } x \neq 3\right\}$ $(-\infty, -2) \cup (-2, 3) \cup (3, +\infty)$?	? $x = -2, x = 3$

(B) Using a graphing calculator, graph each of the functions from Step A, and check to see if vertical asymptotes occur where you expect. Are there any unexpected results? ?

The graphs of the first three functions have the expected vertical asymptotes. The graph of the fourth function, however, has a vertical asymptote at $x = -2$ but not at $x = 3$.

Professional Development

Learning Progressions

Previously, students learned to graph, find asymptotes, and describe domain, range, and end behavior of rational functions of the form $f(x) = \frac{ax+b}{cx+d}$. They used long division to write linear functions divided by linear functions as a quotient plus a remainder, learned the basics about domain, range, and end behavior, and transformed the parent function $f(x) = \frac{1}{x}$ to graph rational functions. Students apply these skills to more complex rational functions such a quadratic divided by linear functions.

Explore

Investigating Domains and Vertical Asymptotes of More Complicated Rational Functions

Avoid Common Errors

Remind students that if the numerator and denominator have the same zero, the result is a point discontinuity, which will not be visible on a graphing calculator. The *x*-value at which the point discontinuity exists must be excluded from the domain, and the corresponding *y*-value must be excluded from the range.

(C) Examine the behavior of $f(x) = \dfrac{x+3}{x-1}$ near $x = 1$.

First, complete the tables.

x approaches 1 from the right	
x	$f(x) = \dfrac{x+3}{x-1}$
1.1	? **41**
1.01	? **401**
1.001	? **4001**

x approaches 1 from the left	
x	$f(x) = \dfrac{x+3}{x-1}$
0.9	? **−39**
0.99	? **−399**
0.999	? **−3999**

Next, summarize the results.

- As $x \to 1^+$, $f(x) \to$? . $+\infty$
- As $x \to 1^-$, $f(x) \to$? . $-\infty$

Based on the behavior of $f(x) = \dfrac{x+3}{x-1}$ near $x = 1$, does the graph of $f(x)$ have a vertical asymptote at $x = 1$? ? **Yes**

(D) Examine the behavior of $f(x) = \dfrac{(x+5)(x-1)}{(x+1)}$ near $x = -1$.

First, complete the tables.

x approaches −1 from the right	
x	$f(x) = \dfrac{(x+5)(x-1)}{x+1}$
−0.9	? **−77.9**
−0.99	? **−797.99**
−0.999	? **−7997.999**

x approaches −1 from the left	
x	$f(x) = \dfrac{(x+5)(x-1)}{x+1}$
−1.1	? **81.9**
−1.01	? **801.99**
−1.001	? **8001.999**

Next, summarize the results.

- As $x \to -1^+$, $f(x) \to$? . $-\infty$
- As $x \to -1^-$, $f(x) \to$? . $+\infty$

Based on the behavior of $f(x) = \dfrac{(x+5)(x-1)}{(x+1)}$ near $x = -1$, does the graph of $f(x)$ have a vertical asymptote at $x = -1$? ? **Yes**

Collaborative Learning

Peer-to-Peer Activity

Have students work in pairs. Give one student in each pair a rational function of the form $f(x) = \dfrac{(x+a)(x+b)}{(x+c)(x+d)}$ where *a*, *b*, *c*, and *d* are integers. Have that student describe the function solely in terms of its zeros, asymptotes, and holes, if any. The partner should attempt to write a rational function that fits this description. Pairs can then discuss their results. Point out that more than one rational function can have the same zeros, asymptotes, and holes.

(E) Examine the behavior of $f(x) = \frac{x-4}{(x+1)(x-1)}$ near $x = -1$ and $x = 1$.

First, complete the tables. Round results to the nearest tenth.

x approaches −1 from the right	
x	$f(x) = \frac{x-4}{(x+1)(x-1)}$
−0.9	? 25.8
−0.99	? 250.8
−0.999	? 2500.8

x approaches −1 from the left	
x	$f(x) = \frac{x-4}{(x+1)(x-1)}$
−1.1	? −24.3
−1.01	? −249.3
−1.001	? −2499.3

x approaches 1 from the right	
x	$f(x) = \frac{x-4}{(x+1)(x-1)}$
1.1	? −13.8
1.01	? −148.8
1.001	? −1498.8

x approaches 1 from the left	
x	$f(x) = \frac{x-4}{(x+1)(x-1)}$
0.9	? 16.3
0.99	? 151.3
0.999	? 1501.3

Next, summarize the results.

- As $x \to -1^+$, $f(x) \to$? . $+\infty$
- As $x \to -1^-$, $f(x) \to$? . $-\infty$
- As $x \to 1^+$, $f(x) \to$? . $-\infty$
- As $x \to 1^-$, $f(x) \to$? . $+\infty$

Based on the behavior of $f(x) = \frac{x-4}{(x+1)(x-1)}$ near $x = -1$, does the graph of $f(x)$ have a vertical asymptote at $x = -1$? ? Yes Based on the behavior of $f(x) = \frac{x-4}{(x+1)(x-1)}$ near $x = 1$, does the graph of $f(x)$ have a vertical asymptote at $x = 1$? ? Yes

(F) Examine the behavior of $f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$ near $x = -2$ and $x = 3$.

First, complete the tables. Round results to the nearest ten thousandth if necessary.

x approaches −2 from the right	
x	$f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$
−1.9	? −8
−1.99	? −98
−1.999	? −998

x approaches −2 from the left	
x	$f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$
−2.1	? 12
−2.01	? 102
−2.001	? 1002

Answers

1. With the function written in the form $f(x) = \frac{(2x+3)(x-3)}{(x+2)(x-3)}$, you can see that the numerator and denominator have a common factor of $x - 3$. In effect, the function behaves just like the function $g(x) = \frac{2x+3}{x+2}$ except that $f(x)$ is not defined at $x = 3$. However, $f(x)$ approaches $g(3) = 1.8$ as x approaches 3.
2. The table feature will display an error message for the function's value at $x = 3$. Because the graph has a "hole" at $x = 3$, there should be an open circle at the point (3, 1.8).

x approaches 3 from the right	
x	$f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$
3.1	? 1.803922
3.01	? 1.800399
3.001	? 1.800040

x approaches 3 from the left	
x	$f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$
2.9	? 1.795918
2.99	? 1.799599
2.999	? 1.799960

Next, summarize the results.

- As $x \to -2^+$, $f(x) \to$? . $-\infty$
- As $x \to -2^-$, $f(x) \to$? . $+\infty$
- As $x \to 3^+$, $f(x) \to$? . 1.8
- As $x \to 3^-$, $f(x) \to$? . 1.8

Based on the behavior of $f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$ near $x = -2$, does the graph of $f(x)$ have a vertical asymptote at $x = -2$? ? **Yes** Based on the behavior of $f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$ near $x = 3$, does the graph of $f(x)$ have a vertical asymptote at $x = 3$? ? **No**

Reflect

1. Rewrite $f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$ so that its numerator and denominator are factored. How does this form of the function explain the behavior of the function near $x = 3$? **See margin.**
2. **Discussion** When you graph $f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$ on a graphing calculator, you can't tell that the function is undefined for $x = 3$. How does using the calculator's table feature help? What do you think the graph should look like to make it clear that the function is undefined at $x = 3$? **See margin.**

Explain 1 Sketching the Graphs of More Complicated Rational Functions

As you have seen, there can be breaks in the graph of a rational function. These breaks are called *discontinuities*, and there are two kinds:

1. When a rational function has a factor in the denominator that is not also in the numerator, an *infinite discontinuity* occurs at the value of x for which the factor equals 0. On the graph of the function, an infinite discontinuity appears as a vertical asymptote.
2. When a rational function has a factor in the denominator that is also in the numerator, a *point discontinuity* occurs at the value of x for which the factor equals 0. On the graph of the function, a point discontinuity appears as a "hole."

The graph of a rational function can also have a horizontal asymptote, or even an asymptote that is a line that is neither horizontal nor vertical. This is determined by the degrees and leading coefficients of the function's numerator and denominator. Examine the following rational expressions, which include polynomial quotients rewritten using long division as a quotient plus a remainder that approaches 0 as x increases or decreases without bound.

Differentiate Instruction

Critical Thinking

Show how to find the slant asymptote for a quadratic/linear function by synthetic division rather than long division.

$\frac{1}{x-1} \longrightarrow 0$ as $x \longrightarrow \pm\infty$

$\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}$ $(x \neq -1)$, so $\frac{x+1}{x^2-1} \longrightarrow 0$ as $x \longrightarrow \pm\infty$

$\frac{x+3}{x-1} = 1 + \frac{4}{x-1} \longrightarrow 1$ as $x \longrightarrow \pm\infty$

$\frac{4x+3}{x-1} = 4 + \frac{7}{x-1} \longrightarrow 4$ as $x \longrightarrow \pm\infty$

$\frac{3x^2+x-1}{x-1} = 3x + 4 + \frac{3}{x-1} \longrightarrow 3x + 4$ as $x \longrightarrow \pm\infty$

In general, if the numerator is a polynomial $p(x)$ in standard form with leading coefficient a and the denominator is a polynomial $q(x)$ in standard form with leading coefficient b, then an examination of the function's end behavior gives the following results.

Relationship between Degree of $p(x)$ and Degree of $q(x)$	Equation of Horizontal Asymptote (if one exists)
Degree of $p(x)$ < degree of $q(x)$	$y = 0$
Degree of $p(x)$ = degree of $q(x)$	$y = \frac{a}{b}$
Degree of $p(x)$ > degree of $q(x)$	There is no horizontal asymptote. The function instead increases or decreases without bound as x increases or decreases without bound. In particular, when the degree of the numerator is 1 more than the degree of the denominator, the function's graph approaches a slanted line, called a *slant asymptote*, as x increases or decreases without bound.

You can sketch the graph of a rational function by identifying where vertical asymptotes, "holes," horizontal asymptotes, and slant asymptotes occur. Using the factors of the numerator and denominator, you can also establish intervals on the x-axis where either an x-intercept or a discontinuity occurs and then check the signs of the factors on those intervals to determine whether the graph lies above or below the x-axis.

Example 1 **Sketch the graph of the given rational function. (If the degree of the numerator is 1 more than the degree of the denominator, find the graph's slant asymptote by dividing the numerator by the denominator.) Also state the function's domain and range using inequalities, set notation, and interval notation. (If your sketch indicates that the function has maximum or minimum values, use a graphing calculator to find those values to the nearest hundredth when determining the range.)**

A $f(x) = \frac{x+1}{x-2}$

Identify vertical asymptotes and "holes."

The function is undefined when $x - 2 = 0$, or $x = 2$. Since $x - 2$ is not a factor of the numerator, there is a vertical asymptote rather than a "hole" at $x = 2$.

Identify horizontal asymptotes and slant asymptotes.

The numerator and denominator have the same degree and the leading coefficient of each is 1, so there is a horizontal asymptote at $y = \frac{1}{1} = 1$.

Explain 1

Sketching the Graphs of More Complicated Rational Functions

Avoid Common Errors

Students may try to graph without factoring and will miss holes and discontinuities. Remind students to factor quadratic expressions when possible. If they determine that the numerator and denominator share a common linear factor, it should be crossed out and the function simplified. There is a hole in the graph at the x-value where the shared linear factor equals zero.

Language Support EL

Vocabulary Development

Have students work in pairs. They should complete a chart like the following, using the different types of rational functions identified in this lesson and the previous lesson.

Rational Function	Example	Vertical Asymptotes	Graph
f(*x*) = *constant/linear*			
f(*x*) = *linear/linear*			
f(*x*) =*quadratic/linear*			
f(*x*) = *quadratic/quadratic*			

Questioning Strategies

How do you find the vertical asymptotes of a rational equation? Simplify the equation by eliminating any factors that are in both the numerator and denominator. There is a linear asymptote at x-values for which the denominator equals zero in the simplified equation.

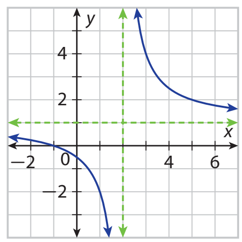
Identify x-intercepts.

An x-intercept occurs when $x + 1 = 0$, or $x = -1$.

Check the sign of the function on the intervals $x < -1$, $-1 < x < 2$, and $x > 2$.

Interval	Sign of $x + 1$	Sign of $x - 2$	Sign of $f(x) = \frac{x+1}{x-2}$
$x < -1$	$-$	$-$	$+$
$-1 < x < 2$	$+$	$-$	$-$
$x > 2$	$+$	$+$	$+$

Sketch the graph using all this information. Then state the domain and range.



Domain:

Inequality: $x < 2$ or $x > 2$

Set notation: $\{x \mid x \neq 2\}$

Interval notation: $(-\infty, 2) \cup (2, +\infty)$

Range:

Inequality: $y < 1$ or $y > 1$

Set notation: $\{y \mid y \neq 1\}$

Interval notation: $(-\infty, 1) \cup (1, +\infty)$

B $f(x) = \dfrac{x^2 + x - 2}{x + 3}$

Factor the function's numerator.

$$f(x) = \frac{x^2 + x - 2}{x + 3} = \frac{(x - 1)(x + 2)}{x + 3}$$

Identify vertical asymptotes and "holes."

The function is undefined when $x + 3 = 0$, or $x = \boxed{-3}$. Since $x + 3$ is not a factor of the numerator, there is a vertical asymptote rather than a "hole" at $x = \boxed{-3}$.

Identify horizontal asymptotes and slant asymptotes.

Because the degree of the numerator is 1 more than the degree of the denominator, there is no horizontal asymptote, but there is a slant asymptote. Divide the numerator by the denominator to identify the slant asymptote.

$$\begin{array}{r}
x - \boxed{2} \\
x + 3 \overline{\smash{\big)}\, x^2 + x - 2} \\
\underline{x^2 + 3x} \\
-2x - 2 \\
\underline{-2x - 6} \\
4
\end{array}$$

So, the line $y = x - \boxed{2}$ is the slant asymptote.

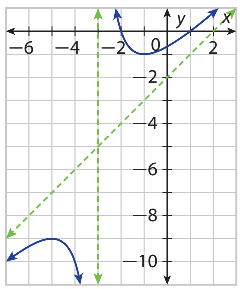
Identify x-intercepts.

There are two x-intercepts: when $x - 1 = 0$, or $x = \boxed{1}$, and when $x + 2 = 0$, or $x = \boxed{-2}$.

Check the sign of the function on the intervals $x < -3$, $-3 < x < -2$, $-2 < x < 1$, and $x > 1$.

Interval	Sign of $x+3$	Sign of $x+2$	Sign of $x-1$	Sign of $f(x) = \frac{(x-1)(x+2)}{x+3}$
$x < -3$	$-$	$-$	$-$	$-$
$-3 < x < -2$	$+$	$-$	$-$	$+$
$-2 < x < 1$	$+$	$+$	$-$	$-$
$x > 1$	$+$	$+$	$+$	$+$

Sketch the graph using all this information. Then state the domain and range.



Domain:

Inequality: $x < \boxed{-3}$ or $x > \boxed{-3}$

Set notation: $\{x \mid x \neq \boxed{-3}\}$

Interval notation: $(-\infty, \boxed{-3}) \cup (\boxed{-3}, +\infty)$

The sketch indicates that the function has a maximum value and a minimum value. Using **3:minimum** from the **CALC** menu on a graphing calculator gives -1 as the minimum value. Using **4:maximum** from the **CALC** menu on a graphing calculator gives -5 as the maximum value.

Range: Inequality: $y < \boxed{-5}$ or $y > \boxed{-1}$

Set notation: $\{y \mid y < \boxed{-5}$ or $y > \boxed{-1}\}$ Interval notation: $(-\infty, \boxed{-5}) \cup (\boxed{-1}, +\infty)$

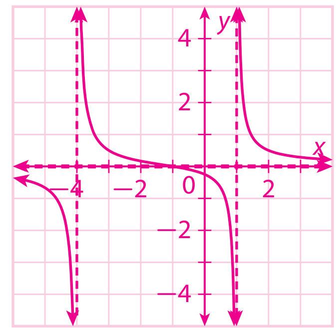
Your Turn

Sketch the graph of the given rational function. (If the degree of the numerator is 1 more than the degree of the denominator, find the graph's slant asymptote by dividing the numerator by the denominator.) Also state the function's domain and range using inequalities, set notation, and interval notation. (If your sketch indicates that the function has maximum or minimum values, use a graphing calculator to find those values to the nearest hundredth when determining the range.)

3. $f(x) = \dfrac{x+1}{x^2 + 3x - 4}$ **See margin.**

Answers

3.



Domain:

$x < -4$ or $-4 < x < 1$ or $x > 1$

$\{x \mid x \neq -4 \text{ and } x \neq 1\}$

$(-\infty, -4) \cup (-4, 1) \cup (1 + \infty)$

Range:

$-\infty < y < +\infty$

$\{y \mid -\infty < y < +\infty\}$

$(-\infty, +\infty)$

Explain 2

Modeling with More Complicated Rational Functions

Avoid Common Errors

Students may not realize when to look for a slant asymptote. Remind students to find the slant asymptote of quadratic/linear rational functions.

Explain 2 Modeling with More Complicated Rational Functions

When two real-world variable quantities are compared using a ratio or rate, the comparison is a rational function. You can solve problems about the ratio or rate by graphing the rational function.

Example 2 **Write a rational function to model the situation, or use the given rational function. State a reasonable domain and range for the function using set notation. Then use a graphing calculator to graph the function and answer the question.**

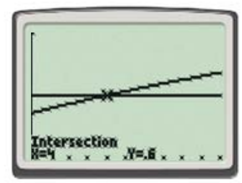
(A) A baseball team has won 32 games out of 56 games played, for a winning percentage of $\frac{32}{56} \approx 0.571$. How many consecutive games must the team win to raise its winning percentage to 0.600?

Let w be the number of consecutive games to be won. Then the total number of games won is the function $T_{\text{won}}(w) = 32 + w$, and the total number of games played is the function $T_{\text{played}}(w) = 56 + w$.

The rational function that gives the team's winning percentage p (as a decimal) is $p(w) = \frac{T_{\text{won}}(w)}{T_{\text{played}}(w)} = \frac{32 + w}{56 + w}$.

The domain of the rational function is $\{w \mid w \geq 0 \text{ and } w \text{ is a whole number}\}$. Note that you do not need to exclude -56 from the domain, because only nonnegative whole-number values of w make sense in this situation.

Since the function models what happens to the team's winning percentage from consecutive wins (no losses), the values of $p(w)$ start at 0.571 and approach 1 as w increases without bound. So, the range is $\{p \mid 0.571 \leq p < 1\}$.



Graph $y = \frac{32 + x}{56 + x}$ on a graphing calculator using a viewing window that shows 0 to 10 on the x-axis and 0.5 to 0.7 on the y-axis. Also graph the line $y = 0.6$. To find where the graphs intersect, select **5: intersect** from the **CALC** menu.

So, the team's winning percentage (as a decimal) will be 0.600 if the team wins 4 consecutive games.

(B) Two friends decide spend an afternoon canoeing on a river. They travel 4 miles upstream and 6 miles downstream. In still water, they know that their average paddling speed is 5 miles per hour. If their canoe trip takes 4 hours, what is the average speed of the river's current? To answer the question, use the rational function $t(c) = \frac{4}{5 - c} + \frac{6}{5 + c} = \frac{50 - 2c}{(5 - c)(5 + c)}$ where c is the average speed of the current (in miles per hour) and t is the time (in hours) spent canoeing 4 miles against the current at a rate of $5 - c$ miles per hour and 6 miles with the current at a rate of $5 + c$ miles per hour.

In order for the friends to travel upstream, the speed of the current must be less than their average paddling speed, so the domain of the function is $\left\{c \mid 0 \le c < \boxed{5}\right\}$. If the friends canoed in still water $(c = 0)$, the trip would take a total of $\frac{4}{5} + \frac{6}{5} = \boxed{2}$ hours.

As c approaches 5 from the left, the value of $\frac{6}{5+c}$ approaches $\frac{6}{10} = 0.6$ hour, but the value of $\frac{4}{5-c}$ increases without bound. So, the range of the function is $\left\{t \mid t \ge \boxed{2}\right\}$.

Graph $y = \dfrac{50 - 2x}{(5-x)(5+x)}$ on a graphing calculator using a viewing window that shows 0 to 5 on the x-axis and 2 to 5 on the y-axis. Also graph the line $y = \boxed{4}$. To find where the graphs intersect, select **5:intersect** from the **CALC** menu. The calculator shows that the average speed of the current is about 3.8 miles per hour.

Your Turn

Write a rational function to model the situation, or use the given rational function. State a reasonable domain and range for the function using set notation. Then use a graphing calculator to graph the function and answer the question.

4. A saline solution is a mixture of salt and water. A $p\%$ saline solution contains $p\%$ salt and $(100 - p)\%$ water by mass. A chemist has 300 grams of a 4% saline solution that needs to be strengthened to a 6% solution by adding salt. How much salt should the chemist add?
The percent p (as a decimal) of the mass of solution that is salt is the rational function $p(s) = \frac{M_{salt}(s)}{M_{solution}(s)} = \frac{12 + s}{300 + s}$. The domain of the rational function is $\left\{s \mid s \ge 0\right\}$. The range is $\left\{p \mid 0.04 \le p < 1\right\}$. The amount of salt that the chemist should add is about 6.4 grams.

Elaborate

5. How can you show that the vertical line $x = c$, where c is a constant, is an asymptote for the graph of a rational function? **See below.**

6. How can you determine the end behavior of a rational function? **See below.**

7. **Essential Question Check-In** How do you identify any vertical asymptotes and "holes" that the graph of a rational function has?
Factor the function's numerator and denominator. If a factor of the denominator is not also a factor of the numerator, then a vertical asymptote occurs at the x-value for which the factor equals 0. If a factor of the denominator is also a factor of the numerator, then a "hole" occurs at the x-value for which the factor equals 0.

5. Examine the behavior of the function as x approaches c from both the left and the right. If the values of the function increase or decrease without bound, the line is an asymptote.

6. Divide the function's numerator by the denominator. The quotient gives the function's end behavior. For instance, if the quotient is a constant c, then the values of the function approach c as x increases or decreases without bound (that is, the horizontal line $y = c$ is an asymptote for the graph).

Questioning Strategies

What are three ways to write the domain and range of a function? set notation, inequalities, interval notation

Elaborate

Integrate Math Processes and Practices

Focus on Generalizing

MPP8 The rules for finding horizontal asymptotes covered in this lesson can be extended to rational functions other than linear/linear, linear/quadratic, quadratic/linear, and quadratic/quadratic functions. When the degree of the numerator is smaller than the degree of the denominator, the horizontal asymptote is $y = 0$. When the degree of the numerator is larger than the degree of the denominator, there is no horizontal asymptote. When the degrees of the numerator and denominator are equal, the horizontal asymptote is $y = \frac{a_1}{a_2}$.

Summarize The Lesson

How do you graph rational functions? Factor if possible, cancel factors common to numerator and denominator, find horizontal, oblique, and vertical asymptotes, plug in x-values to find points, and remember to graph a hole (or point discontinuity) where a cancelled factor equals zero.

Evaluate

Assignment Guide

Level	Concepts and Skills	Practice
Basic	Explore	Exercise 1
	Example 1	Exercises 7–10
	Example 2	Exercises 13, 15
	H.O.T.	Exercise 18
Average	Explore	Exercise 2
	Example 1	Exercises 9–11
	Example 2	Exercises 13–15
	H.O.T.	Exercise 18
Advanced	Explore	Exercise 2
	Example 1	Exercises 10–12
	Example 2	Exercises 14–16
	H.O.T.	Exercises 18–19

Real World Problems

Integrate Math Processes and Practices
Focus on Problem Solving

MPP1 A function in the form $f(x) = \frac{a}{x-h} + k$ can be graphed by transforming the parent function $f(x) = \frac{1}{x}$. It can also be rewritten as a linear/linear rational function by finding a common denominator and rewriting the function as a single fraction.

Avoid Common Errors

Students may assume that when quadratic/linear functions cannot be simplified by cancelling a factor common to the numerator and denominator, that no asymptote exists. Remind students to find the oblique asymptote by dividing the numerator by the denominator. The quotient is the oblique asymptote.

Evaluate: Homework and Practice

Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

State the domain using an inequality, set notation, and interval notation. For any x-value excluded from the domain, state whether the graph has a vertical asymptote or a "hole" at that x-value. Use a graphing calculator to check your answer. **1–2. See below**

1. $f(x) = \frac{x+5}{x+1}$

2. $f(x) = \frac{x^2+2x-3}{x^2-4x+3}$

Divide the numerator by the denominator to write the function in the form $f(x) = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$ and determine the function's end behavior. Then, using a graphing calculator to examine the function's graph, state the range using an inequality, set notation, and interval notation. **3–6. See below**

3. $f(x) = \frac{3x+1}{x-2}$

4. $f(x) = \frac{x}{(x-2)(x+3)}$

5. $f(x) = \frac{x^2-5x+6}{x-1}$

6. $f(x) = \frac{4x^2-1}{x^2+x-2}$

Sketch the graph of the given rational function. (If the degree of the numerator is 1 more than the degree of the denominator, find the graph's slant asymptote by dividing the numerator by the denominator.) Also state the function's domain and range using inequalities, set notation, and interval notation. (If your sketch indicates that the function has maximum or minimum values, use a graphing calculator to find those values to the nearest hundredth when determining the range.) **7–12. Additional Answers**

7. $f(x) = \frac{x-1}{x+1}$

8. $f(x) = \frac{x-1}{(x-2)(x+3)}$

9. $f(x) = \frac{(x+1)(x-1)}{x+2}$

10. $f(x) = \frac{-3x(x-2)}{(x-2)(x+2)}$

11. $f(x) = \frac{x^2+2x-8}{x-1}$

12. $f(x) = \frac{2x^2-4x}{x^2+4x+4}$

1. $x < -1$ or $x > -1$
$\{x \mid x \neq -1\}$
$(-\infty, -1) \cup (-1, +\infty)$
vertical asymptote at $x = -1$

2. $x < 1$ or $1 < x < 3$ or $x > 3$
$\{x \mid x \neq 1 \text{ and } x \neq 3\}$
$(-\infty, 1) \cup (1, 3) \cup (3, +\infty)$
a "hole" at $x = 1$; a vertical asymptote at $x = 3$

3. So, $f(x) = 3 + \frac{7}{x-2}$. As x increases or decreases without bound, $f(x)$ approaches 3.
Range:
$y < 3$ or $y > 3$
$\{y \mid y \neq 3\}$
$(-\infty, 3) \cup (3, +\infty)$

4. So, $f(x) = 0 + \frac{x}{(x-2)(x+3)}$. As x increases or decreases without bound, $f(x)$ approaches 0.
Range:
$-\infty < y < -\infty$
$\{y \mid -\infty < y < -\infty\}$
$(-\infty, +\infty)$

5. So, $f(x) = x - 4 + \frac{2}{x-1}$. As x increases without bound, $f(x)$ increases without bound, and as x decreases without bound, $f(x)$ decreases without bound.
Range (approximate):
$y < -5.83$ or $y > -0.17$
$\{y \mid y < -5.83 \text{ or } y > -0.17\}$
$(-\infty, -5.83) \cup (-0.17, +\infty)$

6. So, $f(x) = 4 + \frac{-4x+7}{x^2+x-2}$. As x increases or decreases without bound, $f(x)$ approaches 4.
Range (approximate):
$y < 0.51$ or $y > 3.49$
$\{y \mid y < 0.51 \text{ or } y > 3.49\}$
$(-\infty, 0.51) \cup (3.49, +\infty)$

Exercise	Depth of Knowledge (D.O.K.)	Math Processes and Practices
1–6	**2** Skills/Concepts	**MPP4** Mathematical Modeling
7–12	**2** Skills/Concepts	**MPP5** Using Mathematical Tools
13–16	**2** Skills/Concepts	**MPP4** Mathematical Modeling
17	**2** Skills/Concepts	**MPP2** Abstract and Quantitative Reasoning
18	**3** Strategic Thinking H.O.T.	**MPP6** Using Precise Mathematical Language
19	**3** Strategic Thinking H.O.T.	**MPP3** Using and Evaluating Logical Reasoning

Write a rational function to model the situation, or use the given rational function. State a reasonable domain and range for the function using set notation. Then use a graphing calculator to graph the function and answer the question.

13. A basketball team has won 16 games out of 23 games played, for a winning percentage (expressed as a decimal) of $\frac{16}{23} \approx 0.696$. How many consecutive games must the team win to raise its winning percentage to 0.750?

The rational function that gives the team's winning percentage p (as a decimal) is $p(w) = \frac{T_{won}(w)}{T_{played}(w)} = \frac{16 + w}{23 + w}$.

The domain of the rational function is $\{w \mid w \geq 0$ and w is a whole number$\}$. Since the values of $p(w)$ start at 0.696 and approach 1 as w increases without bound, the range is $\{p \mid 0.696 \leq p < 1\}$.

14. So far this season, a baseball player has had 84 hits in 294 times at bat, for a batting average of $\frac{84}{294} \approx 0.286$. How many consecutive hits must the player get to raise his batting average to 0.300? **See margin.**

15. A kayaker traveled 5 miles upstream and then 8 miles downstream on a river. The average speed of the current was 3 miles per hour. If the kayaker was paddling for 5 hours, what was the kayaker's average paddling speed? To answer the question, use the rational function $t(s) = \frac{5}{s-3} + \frac{8}{s+3} = \frac{13s - 9}{(s-3)(s+3)}$ where s is the kayaker's average paddling speed (in miles per hour) and t is the time (in hours) spent kayaking 5 miles against the current at a rate of $s - 3$ miles per hour and 8 miles with the current at a rate of $s + 3$ miles per hour. **See margin.**

16. In aviation, *air speed* refers to a plane's speed in still air. A small plane maintains a certain air speed when flying to and from a city that is 400 miles away. On the trip out, the plane flies against a wind, which has an average speed of 40 miles per hour. On the return trip, the plane flies with the wind. If the total flight time for the round trip is 3.5 hours, what is the plane's average air speed? To answer this question, use the rational function $t(s) = \frac{400}{s-40} + \frac{400}{s+40} = \frac{800s}{(s-40)(s+40)}$, where s is the air speed (in miles per hour) and t is the total flight time (in hours) for the round trip.

In order for the plane to make progress when flying against the wind, the plane's air speed must be greater than the speed of the wind, so the domain of the function is $\{s \mid s > 40\}$. As s approaches 40 from the right, $t(s)$ increases without bound, and as s increases without bound, $t(s)$ approaches 0. So, the range of the function is $\{t \mid t > 0\}$.

The plane's average air speed is about 235 miles per hour.

Avoid Common Errors

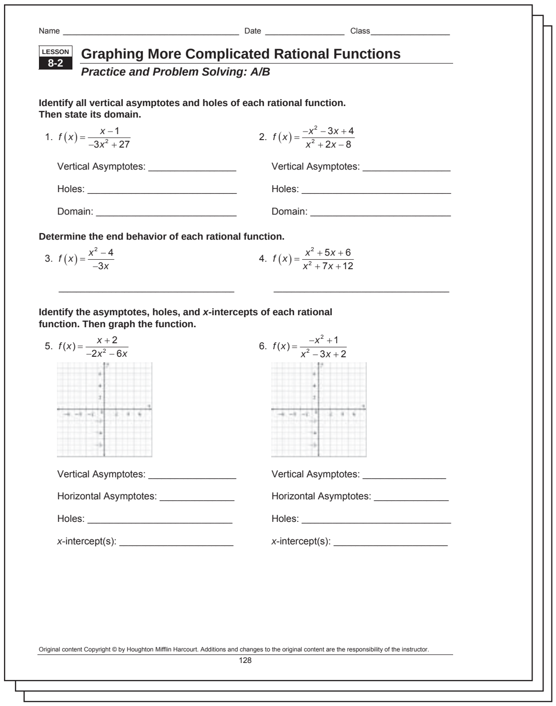
Students may try to graph first and find asymptotes afterward. Remind students to always start by identifying vertical and horizontal asymptotes, before graphing a rational function. The asymptotes are guidelines, or a framework, for the graph.

Journal

Have students outline the process of graphing more complex rational functions in their journals. Suggest that they incorporate several examples.

Online Resources

- Practice and Problem Solving (three forms)
- Reteach
- Reading Strategies
- Success for English Learners



Name ______ Date ______ Class ______

LESSON 8-2 **Graphing More Complicated Rational Functions**

Practice and Problem Solving: A/B

Identify all vertical asymptotes and holes of each rational function. Then state its domain.

1. $f(x) = \frac{x-1}{-3x^2+27}$

Vertical Asymptotes: ______

Holes: ______

Domain: ______

2. $f(x) = \frac{-x^2-3x+4}{x^2+2x-8}$

Vertical Asymptotes: ______

Holes: ______

Domain: ______

Determine the end behavior of each rational function.

3. $f(x) = \frac{x^2-4}{-3x}$

______

4. $f(x) = \frac{x^2+5x+6}{x^2+7x+12}$

______

Identify the asymptotes, holes, and x-intercepts of each rational function. Then graph the function.

5. $f(x) = \frac{x+2}{-2x^2-6x}$

Vertical Asymptotes: ______

Horizontal Asymptotes: ______

Holes: ______

x-intercept(s): ______

6. $f(x) = \frac{-x^2+1}{x^2-3x+2}$

Vertical Asymptotes: ______

Horizontal Asymptotes: ______

Holes: ______

x-intercept(s): ______

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Answers

14. The rational function that gives the player's batting average a (as a decimal) is $a(h) = \frac{T_{hits}(h)}{T_{at\ bats}(h)} = \frac{84 + h}{294 + h}$. The domain of the rational function is $\{h \mid h \geq 0$ and h is a whole number$\}$. Since the values of $a(h)$ start at 0.286 and approach 1 as h increases without bound, the range is $\{a \mid 0.286 \leq a < 1\}$. The number of consecutive hits needed to bring the player's batting average to 0.300 is 6.

15. In order for the kayaker to travel upstream, the paddling speed must exceed the speed of the current, so the domain of the function is $\{s \mid s > 3\}$. As s approaches 3 from the right, $t(s)$ increases without bound, and as s increases without bound, $t(s)$ approaches 0. So, the range of the function is $\{t \mid t > 0\}$. The kayaker's paddling speed was about 4.3 miles per hour.

Avoid Common Errors

Students may set the ratio of volumes to 92.06 instead of 0.9206. Have students explain what a percentage is, and ask them what 92.06% means in this situation. Have students graph $y = 92.06$ and $y = 0.9206$, and ask them which one gives a solution for this problem.

18. The graph of $f(x)$ has a "hole" if the numerator is a factor of the denominator. Since $x^2 + 4x + 3 = (x + 1)(x + 3)$, there are two values of a for which the graph of $f(x)$ has a "hole": $a = 1$ and $a = 3$. The domain of the function $f(x) = \frac{x+1}{x^2+4x+3}$ is $(-\infty, -3) \cup (-3, -1) \cup (-1, +\infty)$ and the range is $(-\infty, 0) \cup \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, +\infty\right)$. The domain of the function $f(x) = \frac{x+3}{x^2+4x+3}$ is $(-\infty, -3) \cup (-3, -1) \cup (-1, +\infty)$, and the range is $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup (0, +\infty)$.

19. The domain of $f(x)$ is $\left\{x \mid x \neq -\frac{1}{2}\right\}$, while the domain of $g(x)$ is $\left\{x \mid x \neq -\frac{1}{2} \text{ and } x \neq \frac{1}{2}\right\}$, so the student's claim is correct with respect to the domains. The range of $f(x)$ is $\{y \mid y \neq -1\}$, while the range of $g(x)$ is $\{y \mid y \neq -1 \text{ and } y \neq 0\}$, so the student's claim is incorrect with respect to the ranges.

17. **Multiple Response** Select the statements that apply to the rational function $f(x) = \frac{x-2}{x^2-x-6}$. **A; D; E; G**

A. The function's domain is $\{x \mid x \neq -2 \text{ and } x \neq 3\}$.

B. The function's domain is $\{x \mid x \neq -2 \text{ and } x \neq -3\}$.

C. The function's range is $\{y \mid y \neq 0\}$.

D. The function's range is $\{y \mid -\infty < y < +\infty\}$.

E. The function's graph has vertical asymptotes at $x = -2$ and $x = 3$.

F. The function's graph has a vertical asymptote at $x = -3$ and a "hole" at $x = 2$.

G. The function's graph has a horizontal asymptote at $y = 0$.

H. The function's graph has a horizontal asymptote at $y = 1$.

H.O.T. Focus on Higher Order Thinking

18. **Draw Conclusions** For what value(s) of a does the graph of $f(x) = \frac{x+a}{x^2+4x+3}$ have a "hole"? Explain. Then, for each value of a, state the domain and the range of $f(x)$ using interval notation.
See margin.

19. **Critique Reasoning** A student claims that the functions $f(x) = \frac{4x^2-1}{4x+2}$ and $g(x) = \frac{4x+2}{4x^2-1}$ have different domains but identical ranges. Which part of the student's claim is correct, and which is false? Explain.
See margin.

Lesson Performance Task

b. $R(r) = \frac{\frac{4}{3}\pi(r - 0.04)^3}{\frac{4}{3}\pi r^3} = \frac{(r-0.04)^3}{r^3}$

In professional baseball, the smallest allowable volume of a baseball is 92.06% of the largest allowable volume, and the range of allowable radii is 0.04 inch.

a. Let r be the largest allowable radius (in inches) of a baseball. Write expressions, both in terms of r, for the largest allowable volume of the baseball and the smallest allowable volume of the baseball. (Use the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$.)
a. Largest allowable volume: $\frac{4}{3}\pi r^3$
Smallest allowable volume: $\frac{4}{3}\pi(r - 0.04)^3$

b. Write and simplify a function that gives the ratio R of the smallest allowable volume of a baseball to the largest allowable volume. **See above.**

c. Use a graphing calculator to graph the function from part b, and use the graph to find the smallest allowable radius and the largest allowable radius of a baseball. Round your answers to the nearest hundredth.
c. Graph $y = \frac{(x-0.04)^3}{x^3}$ and $y = 0.9206$ on a graphing calculator, and find the x-coordinate of the point where the graphs intersect. The graphs intersect at $x = 1.47$ to the nearest hundredth, so the largest allowable radius is 1.47 inches. The smallest allowable radius is $1.47 - 0.04 = 1.43$ inches.

Extension Activity

Have students discuss whether the solution graphs would be different if the baseball were a cube with a side of length x. Have students determine the largest and smallest allowable lengths for the side of the cube when the smallest allowable volume is 92.06% of the largest allowable volume, and the range of allowable lengths is 0.04 inches. Have students compare the function for this situation to the one they derived in the Performance Task.

Lesson Performance Task Scoring Rubric

Points	Criteria
2	Student correctly solves the problem and explains his/her reasoning.
1	Student shows good understanding of the problem but does not fully solve or explain his/her reasoning.
0	Student does not demonstrate understanding of the problem.