8.1 Graphing Simple Rational Functions

Resource Locker

Essential Question: How are the graphs of $f(x) = a\left(\frac{1}{x-h}\right) + k$ and $f(x) = \frac{1}{\frac{1}{b}(x-h)} + k$ related to the graph of $f(x) = \frac{1}{x}$?

Explore 1 Graphing and Analyzing $f(x) = \frac{1}{x}$

A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, where $q(x) \neq 0$. The most basic rational function with a variable expression in the denominator is $f(x) = \frac{1}{x}$.

(A) State the domain of $f(x) = \frac{1}{x}$.

The function accepts all real numbers except ? **0**, because division by ? **0** is undefined. So, the function's domain is as follows:

- As an inequality: $x <$? **0** or $x >$? **0**
- In set notation: $\{x \mid x \neq$? **0** $\}$
- In interval notation (where the symbol $\cup$ means *union*):

$(-\infty,$? **0** $) \cup ($? **0** $, +\infty)$

(B) Determine the end behavior of $f(x) = \frac{1}{x}$.

First, complete the tables.

x Increases without Bound	
x	$f(x) = \frac{1}{x}$
100	? **0.01**
1000	? **0.001**
10,000	? **0.0001**

x Decreases without Bound	
x	$f(x) = \frac{1}{x}$
−100	? **−0.01**
−1000	? **−0.001**
−10,000	? **−0.0001**

Next, summarize the results.

- As $x \to +\infty$, $f(x) \to$? **0**.
- As $x \to -\infty$, $f(x) \to$? **0**.

Professional Development

Learning Progressions

Previously, students looked at polynomial functions and their transformations, inverse variation applications, and the graphs of rational functions of the form $f(x) = \frac{a}{x}$ in the first quadrant. Students now extend that knowledge by graphing and transforming the graph of the parent function $f(x) = \frac{1}{x}$. They learn how the values of h and k are related to the graph of $f(x) = \frac{a}{x-h} + k$ and graph rational functions in the general form $\frac{ax+b}{cx+d}$, where $ax + b$ and $cx + d$ are both linear functions. In the next lesson, students examine the quotients of linear and quadratic polynomials, and discontinuities.

Graphing Simple Rational Functions

Learning Objective

Students will graph simple rational functions, write rational functions to represent a given graph, and model problem situations with rational functions.

Math Processes and Practices

MPP4 Mathematical Modeling

Language Objective

Explain to a partner the characteristics of the graphs of rational functions.

Online Resources

An extra example for each Explain section is available online.

Engage

Essential Question: How are the graphs of $f(x) = a\left(\frac{1}{x-h}\right) + k$ and $f(x) = \frac{1}{\frac{1}{b}(x-h)} + k$ related to the graph of $f(x) = \frac{1}{x}$?

Possible answer: Both graphs involve transformations of the graph of $f(x) = \frac{1}{x}$. The first equation involves vertically stretching or compressing the graph of $f(x) = \frac{1}{x}$ by a factor of a, reflecting it across the x-axis if $a < 0$, and translating it h units horizontally and k units vertically. The second equation involves horizontally stretching or compressing the graph of $f(x) = \frac{1}{x}$ by a factor of b, reflecting it across the y-axis if $b < 0$, and translating it h units horizontally and k units vertically.

Preview: Lesson Performance Task

View the Engage section online. Discuss the photo and how a one-time fee can be converted to a monthly fee. Then preview the Lesson Performance Task.

Explore

Graphing and Analyzing $f(x) = \frac{1}{x}$

Integrate Technology

Students may complete the Explore activity either in the book or online.

Questioning Strategies

Why are there two parts to the graph of the rational function? There are positive values and negative values for *x*, but no value when *x* is 0.

Why isn't the graph symmetric across the *y*-axis? Positive values of *x* result in positive values of *y*, but negative values of *x* result in negative values of *y*.

(C) Be more precise about the end behavior of $f(x) = \frac{1}{x}$, and determine what this means for the graph of the function.

You can be more precise about the end behavior by using the notation $f(x) \to 0^+$, which means that the value of $f(x)$ approaches 0 from the positive direction (that is, the value of $f(x)$ is positive as it approaches 0), and the notation $f(x) \to 0^-$, which means that the value of $f(x)$ approaches 0 from the negative direction. So, the end behavior of the function is more precisely summarized as follows:

- As $x \to +\infty$, $f(x) \to$? . 0^+
- As $x \to -\infty$, $f(x) \to$? . 0^-

The end behavior indicates that the graph of $f(x)$ approaches, but does not cross, the ? , so that axis is an asymptote for the graph. **x-axis**

(D) Examine the behavior of $f(x) = \frac{1}{x}$ near $x = 0$, and determine what this means for the graph of the function.

First, complete the tables.

x Approaches 0 from the Positive Direction	
x	$f(x) = \frac{1}{x}$
0.01	? **100**
0.001	? **1000**
0.0001	? **10,000**

x Approaches 0 from the Negative Direction	
x	$f(x) = \frac{1}{x}$
−0.01	? **−100**
−0.001	? **−1000**
−0.0001	? **−10,000**

Next, summarize the results.

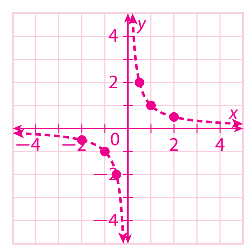
- As $x \to 0^+$, $f(x) \to$? . $+\infty$
- As $x \to 0^-$, $f(x) \to$? . $-\infty$

The behavior of $f(x) = \frac{1}{x}$ near $x = 0$ indicates that the graph of $f(x)$ approaches, but does not cross, the ? , so that axis is also an asymptote for the graph. **y-axis**

(E) Graph $f(x) = \frac{1}{x}$.

First, determine the sign of $f(x)$ on the two parts of its domain.

- When *x* is a negative number, $f(x)$ is a ? **negative** number.
- When *x* is a positive number, $f(x)$ is a ? **positive** number.



Collaborative Learning

Peer-to-Peer Activity

Have students work in pairs. Give each pair graphs of eight transformed rational functions labeled A through H, and cards with the eight functions on them. Set aside one additional graph and function that do not match. Have each pair match the correct functions and graphs and then decide why the remaining graph and function do not match.

Next, complete the tables.

Negative Values of x	
x	$f(x) = \frac{1}{x}$
−2	? -0.5
−1	? -1
−0.5	? -2

Positive Values of x	
x	$f(x) = \frac{1}{x}$
0.5	? 2
1	? 1
2	? 0.5

Finally, use the information from this step and all previous steps to draw the graph. Draw asymptotes as dashed lines.

(F) State the range of $f(x) = \frac{1}{x}$.

The function takes on all real numbers except ? 0, so the function's range is as follows:

- As an inequality: $y <$? 0 or $y >$? 0
- In set notation: $\{y \mid y \neq$? $0\}$
- In interval notation (where the symbol ∪ means *union*): $(-\infty,$? $0) \cup ($? $0, +\infty)$

(G) Identify the intervals where the function is increasing and where it is decreasing. ? **See margin.**

(H) Determine whether $f(x) = \frac{1}{x}$ is an even function, an odd function, or neither. ?

$f(x) = \frac{1}{x}$ **is an odd function.**

Reflect

1. The graph of the function never crosses the x-axis.

1. How does the graph of $f(x) = \frac{1}{x}$ show that the function has no zeros?
2. **Discussion** A graph is said to be *symmetric about the origin* (and the origin is called the graph's *point of symmetry*) if for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. Is the graph of $f(x) = \frac{1}{x}$ symmetric about the origin? Explain. **See margin.**
3. Give any line(s) of symmetry for the graph of $f(x) = \frac{1}{x}$. **The lines $y = x$ and $y = -x$.**

Explain 1 Graphing Simple Rational Functions

When graphing transformations of $f(x) = \frac{1}{x}$, it helps to consider the effect of the transformations on the following features of the graph of $f(x)$: the vertical asymptote, $x = 0$; the horizontal asymptote, $y = 0$; and two reference points, $(-1, -1)$ and $(1, 1)$. The table lists these features of the graph of $f(x)$ and the corresponding features of the graph of $g(x) = a\left(\frac{1}{\frac{1}{b}(x-h)}\right) + k$. Note that the asymptotes are affected only by the parameters h and k, while the reference points are affected by all four parameters.

Feature	$f(x) = \frac{1}{x}$	$g(x) = a\left(\frac{1}{\frac{1}{b}(x-h)}\right) + k$
Vertical asymptote	$x = 0$	$x = h$
Horizontal asymptote	$y = 0$	$y = k$
Reference point	$(-1, -1)$	$(-b + h, -a + k)$
Reference point	$(1, 1)$	$(b + h, a + k)$

Differentiate Instruction

Multiple Representations

Interpersonal learners may benefit from working in pairs, with one student graphing a quotient of linear functions by rewriting the function in graphing form and the other graphing the function by making a table of values and plotting points. Students should compare the strategies by identifying at least one advantage and one disadvantage for each method.

Explain 1

Graphing Simple Rational Functions

Questioning Strategies

What is the midpoint of the x-coordinates of the two reference points? (h, k)

because $\frac{(-b + h) + (b + h)}{2} = \frac{2h}{2} = h$

and $\frac{(-a + k) + (a + k)}{2} = \frac{2k}{2} = k$

Answers

G. The function is never increasing. The function is decreasing on the intervals $(-\infty, 0)$ and $(0, +\infty)$.

2. Yes, because for every point $(x, y) = \left(x, \frac{1}{x}\right)$ on the graph of the function, the point $\left(-x, \frac{1}{-x}\right) = \left(-x, -\frac{1}{x}\right) = (-x, -y)$ is also on the graph.

Avoid Common Errors

Students may make errors when finding a reference point. Have students check them by confirming that the midpoint of the segment joining them is (h,k).

Example 1 **Identify the transformations of the graph of $f(x) = \frac{1}{x}$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptotes $x = 0$ and $y = 0$ to the reference points $(-1, -1)$ and $(1, 1)$. Also state the domain and range of $g(x)$ using inequalities, set notation, and interval notation.**

(A) $g(x) = 3\left(\frac{1}{x-1}\right) + 2$

The transformations of the graph of $f(x)$ that produce the graph of $g(x)$ are:

- a vertical stretch by a factor of 3
- a translation of 1 unit to the right and 2 units up

Note that the translation of 1 unit to the right affects only the x-coordinates, while the vertical stretch by a factor of 3 and the translation of 2 units up affect only the y-coordinates.

Feature	$f(x) = \frac{1}{x}$	$g(x) = 3\left(\frac{1}{x-1}\right) + 2$
Vertical asymptote	$x = 0$	$x = 1$
Horizontal asymptote	$y = 0$	$y = 2$
Reference point	$(-1, -1)$	$(-1 + 1, 3(-1) + 2) = (0, -1)$
Reference point	$(1, 1)$	$(1 + 1, 3(1) + 2) = (2, 5)$

Domain of $g(x)$:

Inequality: $x < 1$ or $x > 1$

Set notation: $\{x \mid x \neq 1\}$

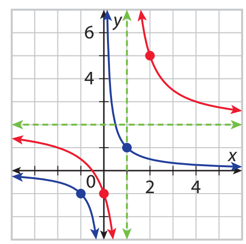
Interval notation: $(-\infty, 1) \cup (1, +\infty)$

Range of $g(x)$:

Inequality: $y < 2$ or $y > 2$

Set notation: $\{y \mid y \neq 2\}$

Interval notation: $(-\infty, 2) \cup (2, +\infty)$



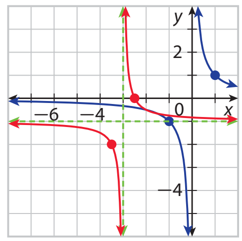
(B) $g(x) = \frac{1}{2(x+3)} - 1$

The transformations of the graph of $f(x)$ that produce the graph of $g(x)$ are:

- a horizontal compression by a factor of $\frac{1}{2}$
- a translation of 3 units to the left and 1 unit down

Note that the horizontal compression by a factor of $\frac{1}{2}$ and the translation of 3 units to the left affect only the x-coordinates of points on the graph of $f(x)$, while the translation of 1 unit down affects only the y-coordinates.

Feature	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{2(x+3)} - 1$
Vertical asymptote	$x = 0$	$x = -3$
Horizontal asymptote	$y = 0$	$y = -1$
Reference point	$(-1, -1)$	$\left(\frac{1}{2}(-1) - 3, -1 - 1\right) = \left(-3\frac{1}{2}, -2\right)$
Reference point	$(1, 1)$	$\left(\frac{1}{2}(1) - 3, 1 - 1\right) = \left(-2\frac{1}{2}, 0\right)$



Domain of $g(x)$:

Inequality: $x < -3$ or $x > -3$

Set notation: $\{x \mid x \neq -3\}$

Interval notation: $(-\infty, -3) \cup (-3, +\infty)$

Range of $g(x)$:

Inequality: $y < -1$ or $y > -1$

Set notation: $\{y \mid y \neq -1\}$

Interval notation: $(-\infty, -1) \cup (-1, +\infty)$

Your Turn

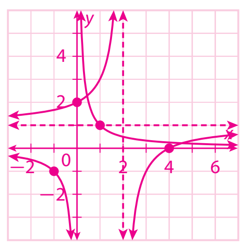
Identify the transformations of the graph of $f(x) = \frac{1}{x}$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptotes $x = 0$ and $y = 0$ to the reference points $(-1, -1)$ and $(1, 1)$. Also state the domain and range of $g(x)$ using inequalities, set notation, and interval notation.

4. $g(x) = -0.5\left(\frac{1}{x+1}\right) - 3$ **See margin.**

5. $g(x) = \frac{1}{-0.5(x-2)} + 1$

The transformations of the graph of $f(x)$ that produce the graph of $g(x)$ are:

- **a horizontal stretch by a factor of 2**
- **a reflection across the y-axis**
- **a translation of 2 units to the right and 1 unit up**



Domain of $g(x)$:

Inequality: $x < 2$ or $x > 2$

Set notation: $\{x \mid x \neq 2\}$

Interval notation: $(-\infty, 2) \cup (2, +\infty)$

Range of $g(x)$:

Inequality: $y < 1$ or $y > 1$

Set notation: $\{y \mid y \neq 1\}$

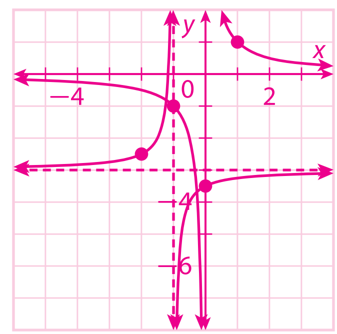
Interval notation: $(-\infty, 1) \cup (1, +\infty)$

Connect Vocabulary EL

Connect the term *rational function* to *rational numbers*, including fractions. Remind students that a simple rational function has polynomials in the numerator and denominator.

Answers

4. The transformations of the graph of $f(x)$ that produce the graph of $g(x)$ are:
   - a vertical compression by a factor of $\frac{1}{2}$
   - a reflection across the x-axis
   - a translation of 1 units to the left and 3 units down



Domain of $g(x)$:

Inequality: $x < -1$ or $x > -1$

Set notation: $\{x \mid x \neq -1\}$

Interval notation: $(-\infty, -1) \cup (-1, +\infty)$

Range of $g(x)$:

Inequality: $y < -3$ or $y > -3$

Set notation: $\{y \mid y \neq -3\}$

Interval notation: $(-\infty, -3) \cup (-3, +\infty)$

Explain 2

Rewriting Simple Rational Functions in Order to Graph Them

Integrate Math Processes and Practices
Focus on Problem Solving

MPP1 Remind students that division by a linear factor in the form $x - h$ can be done by synthetic division. However, for simple quotients, it is quicker to use long division.

Avoid Common Errors

Students may not get the correct sign of the remainder when performing long division.
Remind them to consider the operation signs. For example, when subtracting $3x - 3$ from $3x - 4$, the result is -1.

Explain 2 Rewriting Simple Rational Functions in Order to Graph Them

When given a rational function of the form $g(x) = \frac{mx + n}{px + q}$, where $m \neq 0$ and $p \neq 0$, you can carry out the division of the numerator by the denominator to write the function in the form $g(x) = a\left(\frac{1}{x - h}\right) + k$ or $g(x) = \frac{1}{\frac{1}{b}(x - h)} + k$ in order to graph it.

Example 2 **Rewrite the function in the form $g(x) = a\left(\frac{1}{x - h}\right) + k$ or $g(x) = \frac{1}{\frac{1}{b}(x - h)} + k$ and graph it. Also state the domain and range using inequalities, set notation, and interval notation.**

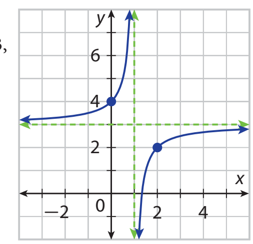
(A) $g(x) = \frac{3x - 4}{x - 1}$

Use long division.

$$\begin{array}{r} 3 \\ x - 1 \overline{)\, 3x - 4} \\ \underline{3x - 3} \\ -1 \end{array}$$

So, the quotient is 3, and the remainder is -1. Using the fact that *dividend* = *quotient* + $\frac{\textit{remainder}}{\textit{divisor}}$, you have $g(x) = 3 + \frac{-1}{x - 1}$, or $g(x) = -\frac{1}{x - 1} + 3$.

The graph of $g(x)$ has vertical asymptote $x = 1$, horizontal asymptote $y = 3$, and reference points $\left(-1 + 1, -(-1) + 3\right) = (0, 4)$ and $(1 + 1, -(1) + 3) = (2, 2)$.



Domain of $g(x)$:
Inequality: $x < 1$ or $x > 1$
Set notation: $\{x | x \neq 1\}$
Interval notation: $(-\infty, 1) \cup (1, +\infty)$

Range of $g(x)$:
Inequality: $y < 3$ or $y > 3$
Set notation: $\{y | y \neq 3\}$
Interval notation: $(-\infty, 3) \cup (3, +\infty)$

(B) $g(x) = \frac{4x - 7}{-2x + 4}$

Use long division.

$$\begin{array}{r} -2 \\ -2x + 4 \overline{)\, 4x - 7} \\ \underline{4x - 8} \\ \boxed{1} \end{array}$$

So, the quotient is -2, and the remainder is 1. Using the fact that *dividend* = *quotient* + $\frac{\textit{remainder}}{\textit{divisor}}$, you have

$$g(x) = -2 + \frac{\boxed{1}}{-2x + 4}, \text{ or } g(x) = \frac{\boxed{1}}{-2\left(x - \boxed{2}\right)} - 2.$$

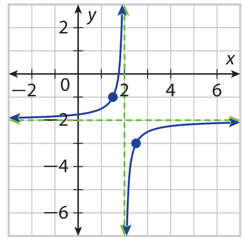
The graph of $g(x)$ has vertical asymptote $x = \boxed{2}$, horizontal asymptote $y = -2$, and reference points $\left(-\frac{1}{2}(-1) + \boxed{2}, -1 - 2\right) = \left(\boxed{2\frac{1}{2}}, -3\right)$ and $\left(-\frac{1}{2}(1) + \boxed{2}, 1 - 2\right) = \left(\boxed{1\frac{1}{2}}, -1\right)$.

Domain of $g(x)$:

Inequality: $x < \boxed{2}$ or $x > \boxed{2}$

Set notation: $\left\{x \mid x \neq \boxed{2}\right\}$

Interval notation: $\left(-\infty, \boxed{2}\right) \cup \left(\boxed{2}, +\infty\right)$



Range of $g(x)$:

Inequality: $y < \boxed{-2}$ or $y > \boxed{-2}$

Set notation: $\left\{y \mid y \neq \boxed{-2}\right\}$

Interval notation: $\left(-\infty, \boxed{-2}\right) \cup \left(\boxed{-2}, +\infty\right)$

Reflect

6. The graph of $g(x)$ is a reflection of the graph of $f(x)$ across the x-axis and a translation 1 unit to the right and 3 units up.

6. In Part A, the graph of $g(x)$ is the result of what transformations of the graph of $f(x) = \frac{1}{x}$?

7. In Part B, the graph of $g(x)$ is the result of what transformations of the graph of $f(x) = \frac{1}{x}$?

The graph of $g(x)$ is a horizontal compression of the graph of $f(x)$ by a factor of $\frac{1}{2}$, a reflection in the y-axis, and a translation 2 units to the right and 2 units down.

Your Turn

Rewrite the function in the form $g(x) = a\left(\frac{1}{x-h}\right) + k$ or $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$ and graph it. Also state the domain and range using inequalities, set notation, and interval notation.

8. $g(x) = \dfrac{3x + 8}{x + 2}$ **See margin.**

Explain 3 Writing Simple Rational Functions

When given the graph of a simple rational function, you can write its equation using one of the general forms $g(x) = a\left(\frac{1}{x-h}\right) + k$ and $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$ after identifying the values of the parameters using information obtained from the graph.

Example 3

Ⓐ **Write the function whose graph is shown. Use the form $g(x) = a\left(\frac{1}{x-h}\right) + k$.**

Since the graph's vertical asymptote is $x = 3$, the value of the parameter h is 3. Since the graph's horizontal asymptote is $y = 4$, the value of the parameter k is 4.

Substitute these values into the general form of the function.

$g(x) = a\left(\frac{1}{x-3}\right) + 4$

Questioning Strategies

Why are the domain and range of the graphs of rational functions disjunctions *or* inequalities, and not conjunctions? The domain does not include the x-asymptote, but it does contain the values of x on each side of the asymptote. This is also true of the range and the y-asymptote.

Explain 3

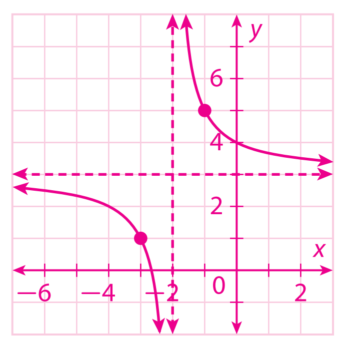
Writing Simple Rational Functions

Questioning Strategies

How do you know that the graph can be modeled by a rational function? There is a vertical asymptote; there is a horizontal asymptote; as x decreases without bound, $f(x)$ approaches the horizontal-asymptote; as x increases without bound, $f(x)$ approaches the horizontal-asymptote; as x approaches the **vertical**-asymptote from the left, $f(x) \to -\infty$; as x approaches the vertical-asymptote from the right, $f(x) \to \infty$.

Answers

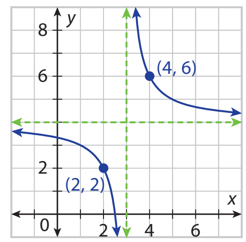
8. $g(x) = 2\left(\frac{1}{x+2}\right) + 3$



Domain of $g(x)$:
Inequality: $x < -2$ or $x > -2$
Set notation: $\left\{x \mid x \neq -2\right\}$
Interval notation: $(-\infty, -2) \cup (-2, +\infty)$

Range of $g(x)$:
Inequality: $y < 3$ or $y > 3$
Set notation: $\left\{y \mid y \neq 3\right\}$
Interval notation: $(-\infty, 3) \cup (3, +\infty)$

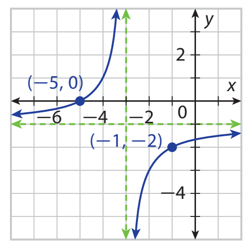
Now use one of the points, such as $(4, 6)$, to find the value of the parameter a.



$$g(x) = a\left(\frac{1}{x-3}\right) + 4$$
$$6 = a\left(\frac{1}{4-3}\right) + 4$$
$$6 = a + 4$$
$$2 = a$$

So, $g(x) = 2\left(\frac{1}{x-3}\right) + 4.$

(B) **Write the function whose graph is shown. Use the form $g(x) = \dfrac{1}{\frac{1}{b}(x-h)} + k$.**



Since the graph's vertical asymptote is $x = -3$, the value of the parameter h is -3. Since the graph's horizontal asymptote is $y = -1$, the value of the parameter k is -1.

Substitute these values into the general form of the function.

$$g(x) = \frac{1}{\frac{1}{b}(x+3)} + -1$$

Now use one of the points, such as $(-5, 0)$, to find the value of the parameter a.

$$g(x) = \frac{1}{\frac{1}{b}(x+3)} + -1$$
$$0 = \frac{1}{\frac{1}{b}(-5+3)} + -1$$
$$1 = \frac{1}{\frac{1}{b}(-2)}$$
$$\frac{1}{b}(-2) \cdot 1 = 1$$
$$\frac{1}{b} = \frac{-1}{2}$$
$$b = -2$$

So, $g(x) = \dfrac{1}{\frac{1}{-2}(x+3)} + -1$, or $g(x) = \dfrac{1}{-0.5(x+3)} - 1.$

Reflect

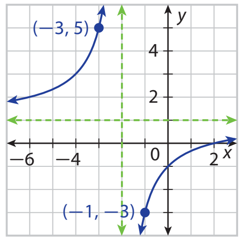
9. **Discussion** In Parts A and B, the coordinates of a second point on the graph of $g(x)$ are given. In what way can those coordinates be useful?

You can use those coordinates as a check on the correctness of the equation by substituting the coordinates into the equation and seeing if you get a true statement.

Your Turn

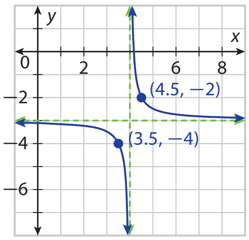
10. Write the function whose graph is shown.
Use the form $g(x) = a\left(\frac{1}{x-h}\right) + k$.

$g(x) = -4\left(\frac{1}{x+2}\right) + 1$



11. Write the function whose graph is shown. Use the form $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$.

$g(x) = \frac{1}{2(x-4)} - 3$



Explain 4 Modeling with Simple Rational Functions

In a real-world situation where there is a shared cost and a per-person or per-item cost, you can model the situation using a rational function that has the general form $f(x) = \frac{a}{x-h} + k$ where $f(x)$ is the total cost for each person or item.

Example 4

(A) **Mary and some of her friends are thinking about renting a car while staying at a beach resort for a vacation. The cost per person for staying at the beach resort is \$300, and the cost of the car rental is \$220. If the friends agree to share the cost of the car rental, what is the minimum number of people who must go on the trip so that the total cost for each person is no more than \$350?**

Analyze Information

Identify the important information.

- The cost per person for the resort is \$300.
- The cost of the car rental is \$220.
- The most that each person will spend is \$350.

Formulate a Plan

Create a rational function that gives the total cost for each person. Graph the function, and use the graph to answer the question.

Explain 4

Modeling with Simple Rational Functions

Questioning Strategies

What indicates that a rational function is a good model for a situation? A constant quantity divided among varying numbers of another quantity is one possibility.

Integrate Math Processes and Practices
Focus on Using Mathematical Tools

MPP5 Students can graph a rational function using a graphing calculator, and use the TRACE function to explore the relationship between the quantities. After TRACE is pressed, the left and right arrow keys move the cursor along the graph. As the cursor moves, its x- and y-coordinates are updated at the bottom of the viewing window.

Elaborate

Integrate Math Processes and Practices
Focus on Using and Evaluating Logical Reasoning

MPP3 Ask how could you graph $C(a) = \frac{a+5}{a+20}$ in the first quadrant without using a graphing calculator. Elicit that you know that the graph begins at $(0, 0.25)$ and that you can determine that the graph has $C(a) = 1$ as its horizontal asymptote (which is the function's end behavior as a increases without bound). You can then make a table of values for several positive values of a, plot the points, and draw a smooth curve that approaches the line $C(a) = 1$ for greater values of a.

Questioning Strategies

For the function $f(x) = \frac{2x-5}{x-4}$, what attributes of the graph can be directly read from the coefficients? $y = 2$ is the horizontal asymptote, $x = 4$ is the vertical asymptote.

Summarize The Lesson

How do you transform the graph of $f(x) = \frac{1}{x}$?
$f(x) = a\left(\frac{1}{\frac{1}{b}x - h}\right) + k$ vertically stretches or compresses the graph of $f(x) = \frac{1}{x}$ by a factor of a and reflects it across the x-axis if $a < 0$, horizontally stretches or compresses the graph by a factor of b and reflects it across the y-axis if $b < 0$, and translates it h units horizontally and k units vertically.

Solve

Let p be the number of people who agree to go on the trip. Let $C(p)$ be the cost (in dollars) for each person.

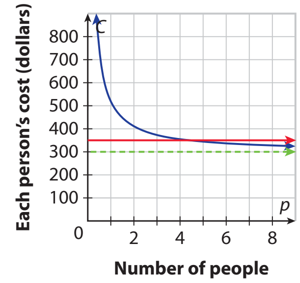
$C(p) = \frac{\boxed{220}}{p} + \boxed{300}$

Graph the function, recognizing that the graph involves two transformations of the graph of the parent rational function:

- a vertical stretch by a factor of 220
- a vertical translation of 300 units up

Also draw the line $C(p) = 350$.

The graphs intersect between $p = \boxed{4}$ and $p = \boxed{5}$, so the minimum number of people who must go on the trip in order for the total cost for each person to be no more than \$350 is 5.



Justify and Evaluate

Check the solution by evaluating the function $C(p)$. Since $C(4) = \boxed{355} > 350$ and $C(5) = \boxed{344} < 350$, the minimum number of people who must go on the trip is 5.

12. Let s be the number of T-shirts. Let $C(s)$ be the average cost (in dollars) of the silk-screened T-shirts when the cost of the kit is included. $C(s) = \frac{200}{s} + 10$. The minimum number of T-shirts that brings the average cost below \$17.50 is 27.

Your Turn

12. Justin has purchased a basic silk screening kit for applying designs to fabric. The kit costs \$200. He plans to buy T-shirts for \$10 each, apply a design that he creates to them, and then sell them. Model this situation with a rational function that gives the average cost of a silk-screened T-shirt when the cost of the kit is included in the calculation. Use the graph of the function to determine the minimum number of T-shirts that brings the average cost below \$17.50. **See Additional Answers for graph.**

Elaborate

13–14. See Additional Answers.

13. Compare and contrast the attributes of $f(x) = \frac{1}{x}$ and the attributes of $g(x) = -\frac{1}{x}$.

14. State the domain and range of $f(x) = a\left(\frac{1}{x-h}\right) + k$ using inequalities, set notation, and interval notation.

15. Given that the model $C(p) = \frac{100}{p} + 50$ represents the total cost C (in dollars) for each person in a group of p people when there is a shared expense and an individual expense, describe what the expressions $\frac{100}{p}$ and 50 represent. **See below.**

16. **Essential Question Check-In** Describe the transformations you must perform on the graph of $f(x) = \frac{1}{x}$ to obtain the graph of $f(x) = a\left(\frac{1}{x-h}\right) + k$. **If $a < 0$, reflect the parent graph across the x-axis. Then either stretch the graph vertically by a factor of $|a|$ if $|a| > 1$ or compress the graph vertically by a factor of $|a|$ if $0 < |a| < 1$. Finally, translate the graph h units right if $h > 0$, $|h|$ units left if $h < 0$, k units up if $k > 0$, and $|k|$ units down if $k < 0$.**

15. The expression $\frac{100}{p}$ represents each person's share of a \$100 expense for the group. The expression 50 represents an individual expense of \$50.

Language Support EL

Communicate Math

Have students work in pairs to look at graphs of rational functions and their equations. The first student describes the graph to the second student, identifying the asymptotes and the domain, range, and end behavior of the function. Students switch roles and repeat the process with a different rational function and its graph.

Evaluate: Homework and Practice

- Online Homework
- Hints and Help
- Extra Practice

Describe how the graph of $g(x)$ is related to the graph of $f(x) = \frac{1}{x}$.

1. $g(x) = \frac{1}{x} + 4$
translation up 4 units

2. $g(x) = 5\left(\frac{1}{x}\right)$
vertical stretch by a factor of 5

3. $g(x) = \frac{1}{x+3}$
translation left 3 units

4. $g(x) = \frac{1}{0.1x}$
horizontal stretch by a factor of 10

5. $g(x) = \frac{1}{x} - 7$
translation down 7 units

6. $g(x) = \frac{1}{x-8}$
translation right 8 units

7. $g(x) = -0.1\left(\frac{1}{x}\right)$
See Additional Answers.

8. $g(x) = \frac{1}{-3x}$
See Additional Answers.

Identify the transformations of the graph of $f(x)$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptotes $x = 0$ and $y = 0$ and to the reference points $(-1, -1)$ and $(1, 1)$. Also state the domain and range of $g(x)$ using inequalities, set notation, and interval notation.

9. $g(x) = 3\left(\frac{1}{x+1}\right) - 2$

10. $g(x) = \frac{1}{-0.5(x-3)} + 1$
See below.

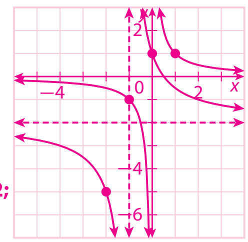
11. $g(x) = -0.5\left(\frac{1}{x-1}\right) - 2$
11. See Additional Answers.

12. $g(x) = \frac{1}{2(x+2)} + 3$

9. The transformations are:
- a vertical stretch by a factor of 3
- a translation of 1 unit to the left and 2 units down

Domain of $g(x)$: $x < -1$ or $x > -1$; $\{x \mid x \neq -1\}$; $(-\infty, -1) \cup (-1, +\infty)$

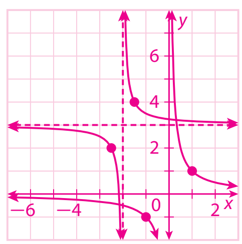
Range of $g(x)$: $y < -2$ or $y > -2$; $\{y \mid y \neq -2\}$; $(-\infty, -2) \cup (-2, +\infty)$



12. The transformations are:
- a horizontal compression by a factor of $\frac{1}{2}$
- a translation of 2 units to the left and 3 units up

Domain of $g(x)$: $x < -2$ or $x > -2$; $\{x \mid x \neq -2\}$; $(-\infty, -2) \cup (-2, +\infty)$

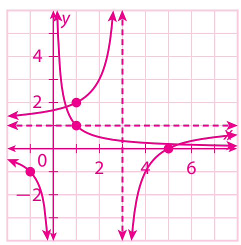
Range of $g(x)$: $y < 3$ or $y > 3$; $\{y \mid y \neq 3\}$; $(-\infty, 3) \cup (3, +\infty)$



10. The transformations are:
- a horizontal stretch by a factor of 2
- a reflection across the y-axis
- a translation of 3 units to the right and 1 unit up

Domain of $g(x)$: $x < 3$ or $x > 3$; $\{x \mid x \neq 3\}$; $(-\infty, 3) \cup (3, +\infty)$

Range of $g(x)$: $y < 1$ or $y > 1$; $\{y \mid y \neq 1\}$; $(-\infty, 1) \cup (1, +\infty)$



Exercise	Depth of Knowledge (D.O.K.)	Math Processes and Practices
1–8	**1** Recall of Information	**MPP2** Abstract and Quantitative Reasoning
9–11	**2** Skills/Concepts	**MPP2** Abstract and Quantitative Reasoning
17–20	**1** Recall of Information	**MPP2** Abstract and Quantitative Reasoning
21	**1** Recall of Information	**MPP4** Mathematical Modeling
22	**2** Skills/Concepts	**MPP4** Mathematical Modeling
23	**2** Skills/Concepts H.O.T.	**MPP5** Using Mathematical Tools
24–25	**3** Strategic Thinking H.O.T.	**MPP3** Using and Evaluating Logical Reasoning

Evaluate

Assignment Guide

Level	Concepts and Skills	Practice
Basic	Explore	N/A
	Example 1	Exercises 1–9, 12
	Example 2	Exercises 13–14
	Example 3	Exercise 17
	Example 4	Exercise 21
	H.O.T.	Exercise 25
Average	Explore	N/A
	Example 1	Exercises 1–10
	Example 2	Exercises 14–15
	Example 3	Exercises 17–18
	Example 4	Exercise 21
	H.O.T.	Exercise 24
Advanced	Explore	N/A
	Example 1	Exercises 10–12
	Example 2	Exercises 14–16
	Example 3	Exercises 18–20
	Example 4	Exercises 21–23
	H.O.T.	Exercises 24–25

Real World Problems

Questioning Strategies

If the coordinates of (h, k) are $(-6, 15)$ and a and b are each equal to 2, what are the coordinates of the reference points? $(-2 + (-6), (-2 + 15) = (-8, 13)$ and $(2 + (-6), (2 + 15) = (-4, 17)$

Integrate Math Processes and Practices Focus on Problem Solving

MPP1 Students may lightly connect the reference points to see that they are symmetric about (h, k).

Connect Vocabulary EL

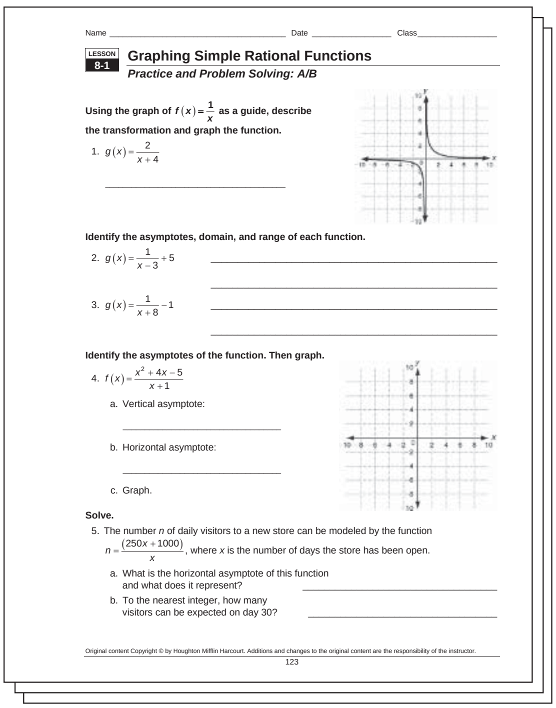
Relate the term *reference point* to the noun *reference*, meaning *a source of information*. A *reference* on a job application is person who serves as a source for information about the applicant.

Avoid Common Errors

For real-world problems, students may not check to see whether the domain is continuous or discrete, and if it is discrete, how a solution should be rounded. Stress the importance of checking whether solutions are defined and whether it is appropriate to round up or down.

Online Resources

- Practice and Problem Solving (three forms)
- Reteach
- Reading Strategies
- Success for English Learners



Name ________ Date ________ Class ________

LESSON 8-1 **Graphing Simple Rational Functions**

Practice and Problem Solving: A/B

Using the graph of $f(x) = \frac{1}{x}$ as a guide, describe the transformation and graph the function.

1. $g(x) = \frac{2}{x+4}$

Identify the asymptotes, domain, and range of each function.

2. $g(x) = \frac{1}{x-3} + 5$

3. $g(x) = \frac{1}{x+8} - 1$

Identify the asymptotes of the function. Then graph.

4. $f(x) = \frac{x^2 + 4x - 5}{x+1}$

a. Vertical asymptote:

b. Horizontal asymptote:

c. Graph.

Solve.

5. The number n of daily visitors to a new store can be modeled by the function $n = \frac{(250x + 1000)}{x}$, where x is the number of days the store has been open.

a. What is the horizontal asymptote of this function and what does it represent?

b. To the nearest integer, how many visitors can be expected on day 30?

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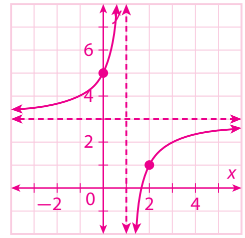
Rewrite the function in the form $g(x) = a\frac{1}{(x-h)} + k$ or $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$ and graph it. Also state the domain and range using inequalities, set notation, and interval notation.

13. $g(x) = \frac{3x-5}{x-1}$

$g(x) = -2\left(\frac{1}{x-1}\right) + 3$

Domain of $g(x)$:
$x < 1$ or $x > 1$
$\{x|x \neq 1\}$
$(-\infty, 1) \cup (1, +\infty)$

Range of $g(x)$:
$y < 3$ or $y > 3$
$\{y|y \neq 3\}$
$(-\infty, 3) \cup (3, +\infty)$



14. $g(x) = \frac{x+5}{0.5x+2}$

See below.

15. $g(x) = \frac{-4x+11}{x-2}$

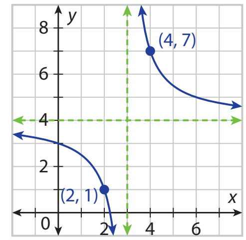
See below.

16. $g(x) = \frac{4x+13}{-2x-6}$

See margin.

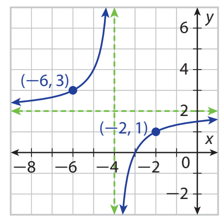
17. Write the function whose graph is shown. Use the form $g(x) = a\left(\frac{1}{x-h}\right) + k$.

$g(x) = 3\left(\frac{1}{x-3}\right) + 4$



18. Write the function whose graph is shown. Use the form $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$

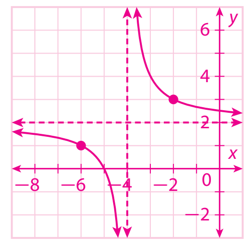
$g(x) = \frac{1}{-\frac{1}{2}(x+4)} + 2$



14. $g(x) = \frac{1}{0.5(x+4)} + 2$

Domain of $g(x)$:
$x < -4$ or $x > -4$
$\{x|x \neq -4\}$
$(-\infty, -4) \cup (-4, +\infty)$

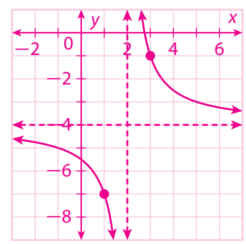
Range of $g(x)$:
$y < 2$ or $y > 2$
$\{y|y \neq 2\}$
$(-\infty, 2) \cup (2, +\infty)$



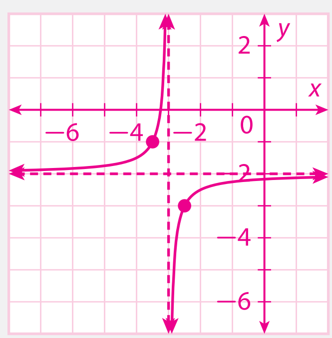
15. $g(x) = 3\left(\frac{1}{x-2}\right) - 4$

Domain of $g(x)$:
$x < 2$ or $x > 2$
$\{x|x \neq 2\}$
$(-\infty, 2) \cup (2, +\infty)$

Range of $g(x)$:
$y < -4$ or $y > -4$
$\{y|y \neq -4\}$
$(-\infty, -4) \cup (-4, +\infty)$



16. $g(x) = \frac{1}{-2(x+3)} - 2$

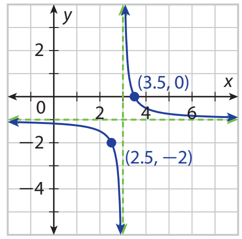


Domain of $g(x)$:
$x < -3$ or $x > -3$
$\{x|x \neq -3\}$
$(-\infty, -3) \cup (-3, +\infty)$

Range of $g(x)$:
$y < -2$ or $y > -2$
$\{y|y \neq -2\}$
$(-\infty, -2) \cup (-2, +\infty)$

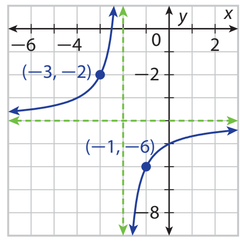
19. Write the function whose graph is shown. Use the form $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$.

$g(x) = \frac{1}{2(x-3)} - 1$



20. Write the function whose graph is shown. Use the form $g(x) = a\left(\frac{1}{x-h}\right) + k$

$g(x) = -2\left(\frac{1}{x+2}\right) - 4$



21. Maria has purchased a basic stained glass kit for \$100. She plans to make stained glass suncatchers and sell them. She estimates that the materials for making each suncatcher will cost \$15. Model this situation with a rational function that gives the average cost of a stained glass suncatcher when the cost of the kit is included in the calculation. Use the graph of the function to determine the minimum number of suncatchers that brings the average cost below \$22.50. **See margin.**

22. Amy has purchased a basic letterpress kit for \$140. She plans to make wedding invitations. She estimates that the cost of the paper and envelope for each invitation is \$2. Model this situation with a rational function that gives the average cost of a wedding invitation when the cost of the kit is included in the calculation. Use the graph of the function to determine the minimum number of invitations that brings the average cost below \$5. **See margin.**

23. **Multiple Response** Select the transformations of the graph of the parent rational function that result in the graph of $g(x) = \frac{1}{2(x-3)} + 1$ **B; E; G**

A. Horizontal stretch by a factor of 2

B. Horizontal compression by a factor of $\frac{1}{2}$

C. Vertical stretch by a factor of 2

D. Vertical compression by a factor of $\frac{1}{2}$

E. Translation 1 unit up

F. Translation 1 unit down

G. Translation 3 units right

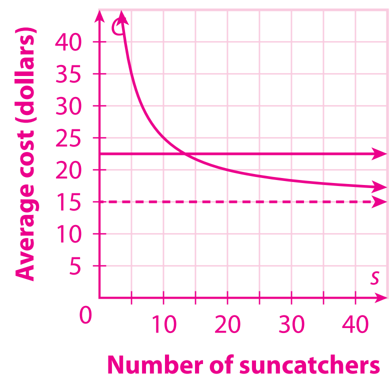
H. Translation 3 units left

Journal

Have students make a table that shows the steps for graphing a function of the form $\frac{ax+b}{cx+d}$ by two methods: (1) writing the function in graphing form before graphing; and (2) graphing the function without writing the function in graphing form first.

Answers

21.

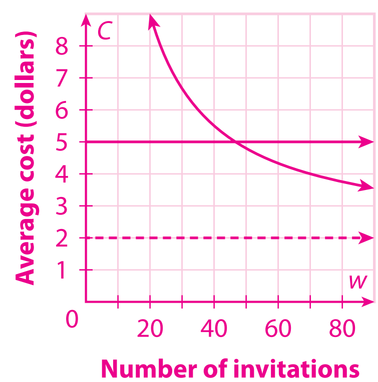


Let s be the number of suncatchers that Maria makes. Let $C(s)$ be the average cost (in dollars) of a suncatcher when the cost of the kit is included.

$C(s) = \frac{100}{s} + 15$

The minimum number of suncatchers that brings the average cost below \$22.50 is 14.

22.



Let w be the number of wedding invitations that Amy makes. Let $C(w)$ be the average cost (in dollars) of a wedding invitation when the cost of the kit is included.

$C(w) = \frac{140}{w} + 2$

The minimum number of invitations that brings the average below \$5 is 47.

Integrate Math Processes and Practices Focus on Using and Evaluating Logical Reasoning

MPP3 Ask students to describe the domain and range of the continuous function $f(x)$ in the Performance Task. Then ask them how the domain and range are different for the real-world situation. Have them discuss how the constraints in the real-world situation affect the domain and range.

Questioning Strategies

What effect does the monthly fee have on $f(x)$? It translates $f(x)$ up or down the y-axis.

As x increases, why does $f(x)$ approach the monthly fee? The initial fee of $240 is spread out over more months, so its monthly contribution becomes smaller and smaller.

Answer

25. The domain consists of all real numbers except for the value of x that makes the denominator equal to 0. Solving $px + q = 0$ for x gives $x = -\frac{q}{p}$. So, the domain is $\left\{x \middle| x \neq -\frac{q}{p}\right\}$. When you divide the numerator by the denominator, the quotient of the leading coefficients determines the horizontal asymptote of the graph, and this y-value is the only real number that isn't in the range of the function. Dividing $mx + n$ by $px + q$ gives a quotient of $\frac{m}{p}$, so the range is $\left\{y \middle| y \neq \frac{m}{p}\right\}$.

Lesson Performance Task Scoring Rubric

Points	Criteria
2	Student correctly solves the problem and explains his/her reasoning.
1	Student shows good understanding of the problem but does not fully solve or explain his/her reasoning.
0	Student does not demonstrate understanding of the problem.

H.O.T. Focus on Higher Order Thinking

24. **Justify Reasoning** Explain why, for positive numbers a and b, a vertical stretch or compression of the graph of $f(x) = \frac{1}{x}$ by a factor of a and, separately, a horizontal stretch or compression of the graph of $f(x)$ by a factor of b result in the same graph when a and b are equal. **See below.**

25. **Communicate Mathematical Ideas** Determine the domain and range of the rational function $g(x) = \frac{mx + n}{px + q}$ where $p \neq 0$. Give your answer in set notation, and explain your reasoning. Assume that there is a remainder when dividing $mx + n$ by $px + q$. **See margin.**

Lesson Performance Task

Graham wants to take snowboarding lessons at a nearby ski resort that charges $40 per week. The resort also charges a one-time equipment-rental fee of $99 for uninterrupted enrollment in classes. The resort requires that learners pay for three weeks of classes at a time.

a. Write a model that gives Graham's average weekly enrollment cost (in dollars) as a function of the time (in weeks) that Graham takes classes.

b. How much would Graham's average weekly enrollment cost be if he took classes only for the minimum of three weeks?

c. For how many weeks would Graham need to take classes for his average weekly enrollment cost to be at most $60? Describe how you can use a graphing calculator to graph the function from part a in order to answer this question, and then state the answer.

a. Let t be the time (in weeks) that Graham takes classes. Let $C(t)$ be Graham's average weekly enrollment cost (in dollars) when the rental fee is included. The model is $C(t) = \frac{99}{t} + 40$.

b. If Graham takes classes for 3 weeks, his average weekly enrollment cost is $C(3) = \frac{99}{3} + 40 = \$73$.

c. To find when Graham's average weekly enrollment cost will be at most $60, graph $y = \frac{99}{x} + 40$ and $y = 60$ on a graphing calculator and find the x-coordinate of the point where the graphs intersect. The graphing calculator gives 4.95 as that x-coordinate. So, since Graham must pay for three weeks of classes at a time, he will need to take classes for 6 weeks in order to have an average weekly enrollment cost of at most $60.

24. A horizontal stretch or compression of the graph of $f(x)$ by a factor of b is given by $f\left(\frac{1}{b}x\right)$, which you can rewrite as follows: $f\left(\frac{1}{b}x\right) = \frac{1}{\frac{1}{b}x} = \frac{1}{\frac{1}{b}} \cdot \frac{1}{x} = b \cdot \frac{1}{x} = b \cdot f(x)$. Since a vertical stretch or compression of the graph of $f(x)$ by a factor of a is given by $a \cdot f(x)$, you can conclude that the two transformations are identical provided $a = b$.

Extension Activity

Ask students to graph the cumulative amount Graham has spent on his membership at the ski resort after x months. Ask them to describe the parent function and tell what translation they need to make. Then have them discuss when Graham might want to use this function and when he would want to use the function in the Performance Task.