2.2 Solving Absolute Value Equations

Essential Question: How can you solve an absolute value equation?

Resource Locker

Explore Solving Absolute Value Equations Graphically

Absolute value equations differ from linear equations in that they may have two solutions. This is indicated with a **disjunction**, a mathematical statement created by a connecting two other statements with the word "or." To see why there can be two solutions, you can solve an absolute value equation using graphs.

(A) Solve the equation $2|x - 5| - 4 = 2$.

Plot the function $f(x) = 2|x - 5| - 4$ on a grid. Then plot the function $g(x) = 2$ as a horizontal line on the same grid, and mark the points where the graphs intersect. **For graph, see Additional Answers.**

The points are (2, 2) and (8, 2).

(B) Write the solution to this equation as a disjunction:

$x =$? or $x =$? **$x = 2$ or $x = 8$**

Reflect

1. Why might you expect most absolute value equations to have two solutions? Why not three or four? **See below.**
2. Is it possible for an absolute value equation to have no solutions? one solution? If so, what would each look like graphically? **See Additional Answers.**

Explain 1 Solving Absolute Value Equations Algebraically

To solve absolute value equations algebraically, first isolate the absolute value expression on one side of the equation the same way you would isolate a variable. Then use the rule:

If $|x| = a$ (where a is a positive number), then $x = a$ OR $x = -a$.

Notice the use of a **disjunction** here in the rule for values of x. You cannot know from the original equation whether the expression inside the absolute value bars is positive or negative, so you must work through both possibilities to finish isolating x.

1. If the absolute value expression is not equal to zero, the expression inside an absolute value can be either positive or negative. So, there can be at most two solutions. Looking at this graphically, an absolute value graph can intersect a horizontal line at most two times.

Professional Development

Integrate Math Processes and Practices

This lesson provides an opportunity to address Math Process and Practice **MPP6**, which calls for students to "attend to precision" and communicate precisely. Students find the solutions to absolute value equations both by graphing them, with and without technology, and through algebra. Students learn that a *disjunction* is often used to express the solutions to absolute value equations, and they use the properties of algebra to accurately and efficiently find the solutions to various types of absolute value equations.

Solving Absolute Value Equations

Learning Objective

Students will solve absolute value equations algebraically and plot the solutions on a number line.

Math Processes and Practices

MPP6 Using Precise Mathematical Language

Language Objective

Explain to a partner why solutions to a variety of absolute value equations make sense and contain more than one solution, one solution, or no solution.

Online Resources

An extra example for each Explain section is available online.

Engage

Essential Question: How can you solve an absolute value equation?

Possible answer: Isolate the absolute value expression, then write two related equations with a disjunction, also known as an "or" statement.

Preview: Lesson Performance Task

View the Engage section online. Discuss the photo and why this situation can be represented by a V-shaped path and an absolute value equation. Then preview the Lesson Performance Task.

Explore

Solving Absolute Value Equations Graphically

Questioning Strategies

How do you solve an absolute value equation graphically? Plot each side as if it were a separate function of *x*, and find the *x*-coordinates of the intersection points.

Explain 1

Solving Absolute Value Equations Algebraically

Avoid Common Errors

Some students may not isolate the absolute value expression on one side of the equation as a first step when solving the equation. Stress the importance of this step so that the equation is in the form $|x| = a$, which has the solution $x = a$ or $x = -a$.

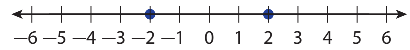
Questioning Strategies

How do you interpret the solutions to an absolute value equation like $|x| = a$ on a number line? Sample answer: The solutions are the same distance from 0 on either side of the number line.

Why is it important to isolate the absolute value expression when solving an absolute value equation? So you can remove the absolute value bars and rewrite the expression as a disjunction.

Example 1 **Solve each absolute value equation algebraically. Graph the solutions on a number line.**

(A) $|3x| + 2 = 8$



Subtract 2 from both sides. $|3x| = 6$

Rewrite as two equations. $3x = 6$ or $3x = -6$

Solve for *x*. $x = 2$ or $x = -2$

(B) $3|4x - 5| - 2 = 19$

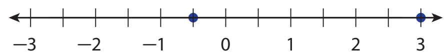
Add 2 to both sides. $3|4x - 5| = 21$

Divide both sides by 3. $|4x - 5| = 7$

Rewrite as two equations. $4x - 5 = 7$ or $4x - 5 = -7$

Add 5 to all four sides. $4x = 12$ or $4x = -2$

Solve for *x*. $x = 3$ or $x = -\frac{1}{2}$

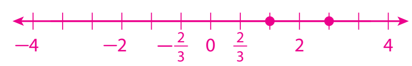


Your Turn

Solve each absolute value equation algebraically. Graph the solutions on a number line.

3. $\frac{1}{2}|x + 2| = 10$ $x = 18$ or $x = -22$

4. $-2|3x - 6| + 5 = 1$ $x = \frac{8}{3}$ or $x = \frac{4}{3}$



Explain 2 Absolute Value Equations with Fewer than Two Solutions

You have seen that absolute value equations have two solutions when the isolated absolute value expression is equal to a positive number. When the absolute value is equal to zero, there is a single solution because zero is its own opposite. When the absolute value expression is equal to a negative number, there is no solution because absolute value is never negative.

Collaborative Learning

Peer-to-Peer Activity

Have students work in pairs to brainstorm types of absolute value equations that have two solutions, one solution, or no solution. For example, instruct one student to write a conjecture about what type of absolute value equation has no solutions, and give an example. Then have the other student solve the example and write an explanation about whether the conjecture is correct or incorrect. Have students switch roles and repeat the exercise using an equation that has a different number of solutions.

Example 2 Isolate the absolute value expression in each equation to determine if the equation can be solved. If so, finish the solution. If not, write "no solution."

(A) $-5|x+1|+2=12$

Subtract 2 from both sides.	$-5|x+1|=10$
Divide both sides by -5.	$|x+1|=-2$
Absolute values are never negative.	No Solution

(B) $\frac{3}{5}|2x-4|-3=-3$

Add 3 to both sides.	$\frac{3}{5}|2x-4|=$ 0
Multiply both sides by $\frac{5}{3}$.	$|2x-4|=$ 0
Rewrite as one equation.	$2x-4=$ 0
Add 4 to both sides.	$2x=$ 4
Divide both sides by 2.	$x=$ 2

8. The range of a non-constant linear function is all real numbers. The range of an absolute value function is $y \geq k$ if the function opens upward and $y \leq k$ if the function opens downward. Because the graph of a linear function is a line, a horizontal line will intersect it only once. Because the graph of an absolute value function is a V, a horizontal line can intersect it once, twice, or not at all.

Your Turn

Isolate the absolute value expression in each equation to determine if the equation can be solved. If so, finish the solution. If not, write "no solution."

5. $3\left|\frac{1}{2}x+5\right|+7=5$ **No solution**

6. $9\left|\frac{4}{3}x-2\right|+7=7$ $x=\frac{3}{2}$

7. The solution to a mathematical equation is not simply any value of the variable that makes the equation true. Supplying only one value that works in the equation implies that it is the only value that works, which is incorrect.

Elaborate

7. Why is important to solve both equations in the disjunction arising from an absolute value equation? Why not just pick one and solve it, knowing the solution for the variable will work when plugged backed into the equation?

8. **Discussion** Discuss how the range of the absolute value function differs from the range of a linear function. Graphically, how does this explain why a linear equation always has exactly one solution while an absolute value equation can have one, two, or no solutions? **See above.**

9. **Essential Question Check-In** Describe, in your own words, the basic steps to solving absolute value equations and how many solutions to expect.

Isolate the absolute value expression. If the absolute value expression is equal to a positive number, solve for both the positive and negative case. If the absolute value expression is equal to zero, then remove the absolute value bars and solve the equation. There is one solution. If the absolute value expression is equal to a negative number, then there is no solution.

Differentiate Instruction

Critical Thinking

Some students may need help in deciding whether absolute value equations have no solutions, one solution, or two solutions. You may want to suggest that they *always* follow this solving plan: (1) Write the original equation; then (2) isolate the absolute value expression on one side of the equal sign. It will have the form $|ax+b|=c$. (3) Rewrite the equation as two equations of the form $ax+b=c$ and $ax+b=-c$; and (4) solve each equation for x. There may be 0, 1, or 2 solutions. (5) If there are two solutions, write the answer using "or." (6) Check the solution(s) in the original problem.

Explain 2

Absolute Value Equations with Fewer than Two Solutions

Questioning Strategies

When does an absolute value equation have fewer than two solutions? when the absolute value expression is equal to 0 or to a negative number

In the absolute value expression $d|ax+b|-c=-c$ for nonzero variables, how does d affect the solution? It does not affect it. The first step is to add c to both sides to get $d|ax+b|=0$. Because the product of a number and 0 is 0, you can divide both sides by d to get $|ax+b|=0$.

Avoid Common Errors

Some students may think that if an absolute value equation does not have two solutions, then there must be no solution. Explain to students that when the absolute value expression equals zero, there will be one solution. For example, $|3x+6|=0$ has one solution, $x=-2$.

Elaborate

Integrate Math Processes and Practices Focus on Generalizing

MPP8 Discuss with students how to solve an absolute value equation of the form $|ax+b|=c$. Students should routinely rewrite the next step as a disjunction, or a compound equation of the form $ax+b=c$ or $ax+b=-c$ and then solve each part of the equation.

Questioning Strategies

How is the process of solving a linear absolute value equation like the process of solving a regular linear equation? Both processes are similar initially, except that you isolate the absolute value in one case, but isolate the variable in the case of the linear equation. From there, the process is the same for each part of the disjunction of the two linear equations for the absolute value equation.

Peer-to-Peer Activity

Have students work in pairs. Have one student write an absolute value equation and have the partner solve it. The partner then explains why the solution(s) makes sense. Students switch roles and repeat the process. Encourage students to use the phrase "distance from zero" and the statement "This negative/positive integer makes the equation true."

Summarize The Lesson

How do you solve a linear absolute value equation? Isolate the absolute value expression; write resulting equation as the disjunction of two linear equations; and solve each equation.

Evaluate

Personal Math Trainer

Assignment Guide

Level	Concepts and Skills	Practice
Basic	Explore	Exercises 1–2
	Example 1	Exercises 5–6
	Example 2	Exercises 9–12, 20
	H.O.T.	Exercise 22
Average	Explore	Exercises 1–4
	Example 1	Exercises 5–8
	Example 2	Exercises 9–12
	H.O.T.	Exercises 22, 25
Advanced	Explore	Exercises 3–4
	Example 1	Exercise 8
	Example 2	Exercises 13–15
	H.O.T.	Exercises 22–25

Real World Problems

Evaluate: Homework and Practice

Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

1–4. For graphs, see Additional Answers.

Solve the following absolute value equations by graphing.

1. $|x-3|+2=5$ **$x=0$ or $x=6$**
2. $2|x+1|+5=9$ **$x=-3$ or $x=1$**
3. $-2|x+5|+4=2$ **$x=-4$ or $x=-6$**
4. $\left|\frac{3}{2}(x-2)\right|+3=2$ **No solution**

Solve each absolute value equation algebraically. Graph the solutions on a number line. 5–8. For graphs, see below.

5. $|2x|=3$ **$x=\frac{3}{2}$ or $x=-\frac{3}{2}$**
6. $\left|\frac{1}{3}x+4\right|=3$ **$x=-3$ or $x=-21$**
7. $3|2x-3|+2=3$ **$x=\frac{5}{3}$ or $x=\frac{4}{3}$**
8. $-8|-x-6|+10=2$ **$x=-7$ or $x=-5$**

Isolate the absolute value expressions in the following equations to determine if they can be solved. If so, find and graph the solution(s). If not, write "no solution". 11–12. For graphs, see Additional Answers.

9. $\frac{1}{4}|x+2|+7=5$ **No solution**
10. $-3|x-3|+3=6$ **No solution**
11. $2\left(|x+4|+3\right)=6$ **$x=-4$**
12. $5|2x+4|-3=-3$ **$x=-2$**

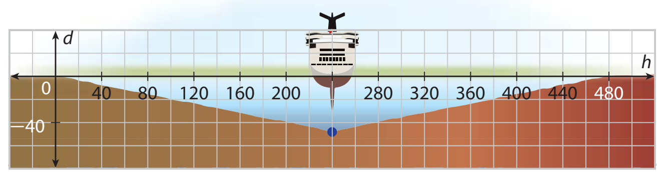
Solve the absolute value equations.

13. $|3x-4|+2=1$ **No solution**
14. $7\left|\frac{1}{2}x+3\frac{1}{2}\right|-2=5$ **$x=-5$ or $x=-9$**
15. $|2(x+5)-3|+2=6$ **$x=-\frac{3}{2}$ or $x=-\frac{11}{2}$**
16. $-5|-3x+2|-2=-2$ **$x=\frac{2}{3}$**

17. The bottom of a river makes a V-shape that can be modeled with the absolute value function, $d(h)=\frac{1}{5}|h-240|-48$, where d is the depth of the river bottom (in feet) and h is the horizontal distance to the left-hand shore (in feet).

A ship risks running aground if the bottom of its keel (its lowest point under the water) reaches down to the river bottom. Suppose you are the harbormaster and you want to place buoys where the river bottom is 30 feet below the surface. How far from the left-hand shore should you place the buoys?

The buoys should be placed at 150 ft and 330 ft from the left-hand shore.



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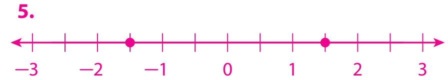

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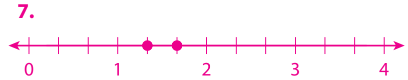

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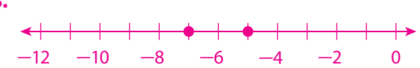
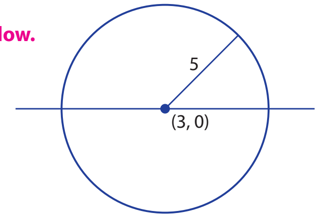


Exercise	Depth of Knowledge (D.O.K.)	Math Processes and Practices
1–4	2 Skills/Concepts	MPP5 Using Mathematical Tools
5–16	2 Skills/Concepts	MPP6 Using Precise Mathematical Language
17	3 Strategic Thinking	MPP4 Mathematical Modeling
18, 21	3 Strategic Thinking	MPP4 Mathematical Modeling
19	3 Strategic Thinking	MPP6 Using Precise Mathematical Language
20	2 Skills/Concepts	MPP6 Using Precise Mathematical Language
22	3 Strategic Thinking H.O.T.	MPP3 Using and Evaluating Logical Reasoning
23–25	3 Strategic Thinking H.O.T.	MPP6 Using Precise Mathematical Language

18. A flock of geese is approaching a photographer, flying in formation. The photographer starts taking photographs when the lead goose is 300 feet horizontally from her, and continues taking photographs until it is 100 feet past. The flock is flying at a steady 30 feet per second. Write and solve an equation to find the times after the photographing begins that the lead goose is at a horizontal distance of 75 feet from the photographer.

The horizontal distance of the lead bird from the photographer is $d(t) = |30t - 300|$.
Solve $|30t - 300| = 75$.
The lead goose is at a horizontal distance of 75 feet from the photographer after 7.5 seconds and again after 12.5 seconds.

19. **Geometry** Find the points where a circle centered at (3, 0) with a radius of 5 crosses the x-axis. Use an absolute value equation and the fact that all points on a circle are the same distance (the radius) from the center. **See below.**



20. Select the value or values of x that satisfy the equation $-\frac{1}{2}|3x - 3| + 2 = 1$.

A. $x = \frac{5}{3}$ **B.** $x = -\frac{5}{3}$

C. $x = \frac{1}{3}$ **D.** $x = -\frac{1}{3}$

E. $x = 3$ **F.** $x = -3$

G. $x = 1$ **H.** $x = -1$

$x = \frac{5}{3}$ or $x = \frac{1}{3}$ (A or C)

21. Terry is trying to place a satellite dish on the roof of his house at the recommended height of 30 feet. His house is 32 feet wide, and the height of the roof can be described by the function $h(x) = -\frac{3}{2}|x - 16| + 24$, where x is the distance along the width of the house. Where should Terry place the dish?

Use the model function to solve for x when $h(x) = 30$ feet.

$$-\frac{3}{2}|x - 16| + 24 = 30$$
$$|x - 16| = -4$$

No Solution. Terry does not have a spot on his roof that is 30 feet high.

19. The points on the x-axis that are a distance of 5 from the center of the circle at $x = 3$ are given by $|x - 3| = 5$.
Solving $|x - 3| = 5$ gives $x = 8$ or $x = -2$.
The points are $(-2, 0)$ and $(8, 0)$.

Avoid Common Errors

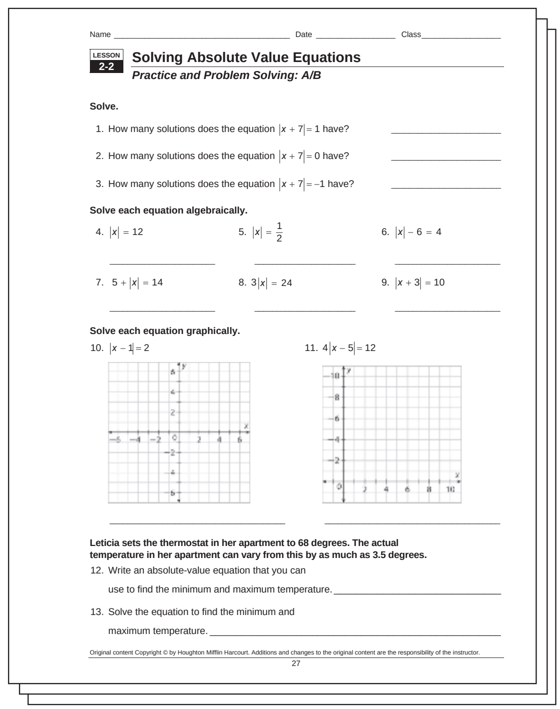
Students may erroneously include points on the graph *between* the solution points when they graph solutions. Remind students that the solution process gives 0, 1, or 2 solutions to an absolute value equation, not infinitely many solutions.

Avoid Common Errors

When solving absolute equations algebraically, watch for students who do not solve these equations by first rewriting them in the form $|ax + b| = c$. Remind them that the absolute value expression should be nonnegative before they proceed with the solution steps.

Online Resources

- Practice and Problem Solving (three forms)
- Reteach
- Reading Strategies
- Success for English Learners



Name ________ Date ________ Class ________

LESSON 2-2 **Solving Absolute Value Equations**
Practice and Problem Solving: A/B

Solve.

1. How many solutions does the equation $|x + 7| = 1$ have? ________
2. How many solutions does the equation $|x + 7| = 0$ have? ________
3. How many solutions does the equation $|x + 7| = -1$ have? ________

Solve each equation algebraically.

4. $|x| = 12$ 5. $|x| = \frac{1}{2}$ 6. $|x| - 6 = 4$

7. $5 + |x| = 14$ 8. $3|x| = 24$ 9. $|x + 3| = 10$

Solve each equation graphically.

10. $|x - 1| = 2$ 11. $4|x - 5| = 12$

Leticia sets the thermostat in her apartment to 68 degrees. The actual temperature in her apartment can vary from this by as much as 3.5 degrees.

12. Write an absolute-value equation that you can use to find the minimum and maximum temperature. ________
13. Solve the equation to find the minimum and maximum temperature. ________

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Avoid Common Errors

Watch for students who are confused by nested absolute value equations. Remind students to carefully write disjunctions for each part of the solution, as appropriate, using the same solution process they use for a single absolute value equation.

Peer-to-Peer Discussion

Ask students to discuss with a partner what the solution to $|ax + b| = c$ means in terms of the graph of the related functions $f(x) = |ax + b|$ and $g(x) = c$. Then ask students to make conjectures about the solutions to $|ax + b| = c$ and the graphs of their related functions. Conjectures should include the possible number of intersection points and how the graph of the function looks. The solutions to $|ax + b| = c$ are the x-coordinates of the intersection points of the related functions. Based on this, conjectures should include that the graphs of $f(x) = |ax + b|$ and $g(x) = c$ can have two, one, or no intersection points, and that the graph of $f(x)$ is V-shaped and this graph can intersect a line in two or fewer places.

Journal

Have students compare and contrast the methods they have learned for solving absolute value equations.

Avoid Common Errors

Some students may use the ratio $\frac{2}{5}$ in their equation instead of $\frac{5}{2}$. Explain that the snowball *rises* 5 feet north for every 2 feet west it *runs*. Thus, the ratio is $\frac{5}{2}$.

Lesson Performance Task Scoring Rubric

Points	Criteria
2	Student correctly solves the problem and explains his/her reasoning.
1	Student shows good understanding of the problem but does not fully solve or explain his/her reasoning.
0	Student does not demonstrate understanding of the problem.

H.O.T. Focus on Higher Order Thinking

22. Explain the Error While attempting to solve the equation $-3|x-4| - 4 = 3$, a student came up with the following results. Explain the error and find the correct solution:

$$-3|x-4| - 4 = 3$$
$$-3|x-4| = 7$$
$$|x-4| = -\frac{7}{3}$$
$$x - 4 = -\frac{7}{3} \quad \text{or} \quad x - 4 = \frac{7}{3}$$
$$x = \frac{5}{3} \quad \text{or} \quad x = \frac{19}{3}$$

The student tried to replace the absolute value equation with two equations using the positive and negative values of the number on the other side of the equal sign. However, this number was negative and cannot be treated like a positive number. The isolated absolute value expression is equal to a negative number and therefore this equation has no solution.

23. Communicate Mathematical Ideas Solve this absolute value equation and explain what algebraic properties make it possible to do so. **See below.**

$$3|x-2| = 5|x-2| - 7$$

24. Justify Your Reasoning This absolute value equation has nested absolute values. Use your knowledge of solving absolute value equations to solve this equation. Justify the number of possible solutions. **See Additional Answers.**

$$\big||2x+5| - 3\big| = 10$$

25. Check for Reasonableness For what type of real-world quantities would the negative answer for an absolute value equation not make sense?
Answers will vary. Sample answer: time, distance, height, length, speed

Lesson Performance Task

A snowball comes apart as a child throws it north, resulting in two halves traveling away from the child. The child is standing 12 feet south and 6 feet east of the school door, along an east-west wall. One fragment flies off to the northeast, moving 2 feet east for every 5 feet north of travel, and the other moves 2 feet west for every 5 feet north of travel. Write an absolute value function that describes the northward position, $n(e)$, of both fragments as a function of how far east of the school door they are. How far apart are the fragments when they strike the wall?

$n(e) = \frac{5}{2}|e - 6|$

The fragments are $\left|\frac{54}{5} - \frac{6}{5}\right| = \left|\frac{48}{5}\right| = 9\frac{3}{5}$ feet apart.

23.

$3|x-2| - 5|x-2| = -7$	**Subtraction Property of Equality**
$(3-5)|x-2| = -7$	**Distributive Property**
$|x-2| = \frac{7}{2}$	**Division Property of Equality**
$x - 2 = \frac{7}{2}$ or $x - 2 = -\frac{7}{2}$	**Definition of absolute value**
$x = \frac{11}{2}$ or $x = -\frac{3}{2}$	**Addition Property of Equality**

Extension Activity

Have students try to find an alternate solution method using the formula for the slope of a line. Student should find the coordinates for the snowball on the right to be $\left(10\frac{4}{5}, 12\right)$. Subtracting 6 from the value of x gives the distance from $(6, 12)$ to $(x, 12)$. That distance is $4\frac{4}{5}$ ft. The distance from the y-axis to the snowball on the left is $6 - 4\frac{4}{5} = 1\frac{1}{5}$ ft. So the fragments are $10\frac{4}{5} - 1\frac{1}{5} = 9\frac{3}{5}$ feet apart.