11.2 Simplifying Radical Expressions

Essential Question: How can you simplify expressions containing rational exponents or radicals involving *n*th roots?

Resource Locker

Explore Establishing the Properties of Rational Exponents

In previous courses, you have used properties of integer exponents to simplify and evaluate expressions, as shown here for a few simple examples:

$$4^2 \cdot 4^3 = 4^{2+3} = 4^5 = 1024 \qquad (4 \cdot x)^2 = 4^2 \cdot x^2 = 16x^2$$

$$(4^2)^3 = 4^{2 \cdot 3} = 4^6 = 4096 \qquad \frac{4^2}{4^3} = 4^{2-3} = 4^{-1} = \frac{1}{4}$$

$$\left(\frac{4}{x}\right)^3 = \frac{4^3}{x^3} = \frac{64}{x^3}$$

Now that you have been introduced to expressions involving rational exponents, you can explore the properties that apply to simplifying them.

(A) Let $a = 64$, $b = 4$, $m = \frac{1}{3}$, and $n = \frac{3}{2}$. Evaluate each expression by substituting and applying exponents individually, as shown.

Expression	Substitute	Simplify	Result
$a^m \cdot a^n$	$64^{\frac{1}{3}} \cdot 64^{\frac{3}{2}}$	$4 \cdot 512$	2048
$(a \cdot b)^n$	$(64 \cdot 4)^{\frac{3}{2}}$	$256^{\frac{3}{2}}$	?
$(a^m)^n$	?	?	?
$\frac{a^n}{a^m}$	?	?	?
$\left(\frac{a}{b}\right)^n$	?	?	?

4096

$\left(64^{\frac{1}{3}}\right)^{\frac{3}{2}}$; $4^{\frac{3}{2}}$; 8

$\frac{64^{\frac{3}{2}}}{64^{\frac{1}{3}}}$; $\frac{512}{4}$; 128

$\left(\frac{64}{4}\right)^{\frac{3}{2}}$; $16^{\frac{3}{2}}$; 64

Professional Development

Learning Progressions

This lesson extends concepts and properties that students have learned in previous courses and lessons. Students are familiar with the properties of exponents, and have used them to simplify expressions containing integer exponents. They also were introduced to the *n*th roots and the meaning of rational exponents in the previous lesson. In this lesson, students combine these concepts, extending the properties to expressions containing rational exponents. They also learn about the properties of *n*th roots. Students will apply these skills in the following lesson, where they will use them to solve radical equations.

Simplifying Radical Expressions

Learning Objective

Students will simplify expressions involving rational exponents and radicals.

Math Processes and Practices

MPP8 Generalizing

Language Objective

Explain to a partner the steps for simplifying rational exponents and radical expressions.

Online Resources

An extra example for each Explain section is available online.

Engage

Essential Question: How can you simplify expressions containing rational exponents or radicals involving nth roots?

Possible answer: You can use the same properties of exponents for rational exponents as for integer exponents, apply the properties of square roots to radicals involving *n*th roots, and translate between radical form and rational exponent form whenever it is helpful.

Preview: Lesson Performance Task

View the Engage section online. Discuss the photo and how the volume and surface area of a sphere have a common variable. Then preview the Lesson Performance Task.

Explore

Establishing the Properties of Rational Exponents

Integrate Technology

Students have the option of completing the Explore activity either in the book or online.

Questioning Strategies

If an expression consists of a variable raised to a negative exponent, how can you rewrite the expression with a positive exponent? Rewrite the expression as the reciprocal of the variable raised to the opposite of the exponent.

How does that help you write the simplified form of $\frac{x^{\frac{1}{4}}}{x^{\frac{3}{4}}}$ with a positive exponent? You can subtract the exponents, and then apply the rule to the answer. $\frac{x^{\frac{1}{4}}}{x^{\frac{3}{4}}} = x^{\frac{1}{4}-\frac{3}{4}} = x^{-\frac{1}{2}} = \left(\frac{1}{x}\right)^{\frac{1}{2}}$

B Complete the table again. This time, however, apply the rule of exponents that you would use for integer exponents.

	Expression	Apply Rule and Substitute	Simplify	Result	
	$a^m \cdot a^n$	$64^{\frac{1}{3}+\frac{3}{2}}$	$64^{\frac{11}{6}}$	?	2048
$64^{\frac{3}{2}} \cdot 4^{\frac{3}{2}}$; $512 \cdot 8$; 4096	$(a \cdot b)^n$	?	?	?	
$64^{\frac{1}{3} \cdot \frac{3}{2}}$; $64^{\frac{1}{2}}$; 8	$(a^m)^n$	?	?	?	
$64^{\frac{3}{2}-\frac{1}{3}}$; $64^{\frac{7}{6}}$; 128	$\frac{a^n}{a^m}$	?	?	?	
$\frac{64^{\frac{3}{2}}}{4^{\frac{3}{2}}}$; $\frac{512}{8}$; 64	$\left(\frac{a}{b}\right)^n$	?	?	?	

Reflect

1. Compare your results in Steps A and B. What can you conclude? **See below.**
2. In Steps A and B, you evaluated $\frac{a^n}{a^m}$ two ways. Now evaluate $\frac{a^m}{a^n}$ two ways, using the definition of negative exponents. Are your results consistent with your previous conclusions about integer and rational exponents?
$\frac{a^m}{a^n} = \frac{64^{\frac{1}{3}}}{64^{\frac{3}{2}}} = \frac{4}{512} = \frac{1}{128}$;
$\frac{a^m}{a^n} = 64^{\frac{1}{3}-\frac{3}{2}} = 64^{-\frac{7}{6}} = \frac{1}{64^{\frac{7}{6}}} = \frac{1}{128}$;
Yes, working with negative rational exponents is consistent with working with negative integer exponents.

1. Applying the same rules as for integer exponents gives the same results as applying the exponents individually. The properties of rational exponents are the same as the corresponding properties of integer exponents.

Collaborative Learning

Peer-to-Peer Activity

Have students work in pairs. Provide each pair with several fairly complex expressions to simplify. Instruct one student in each pair to simplify one of the expressions while the other gives verbal instructions for each step. Then have the student who simplified the expression write an explanation next to each step, describing what was done. Have students switch roles and repeat the exercise using a different expression.

Explain 1 Simplifying Rational-Exponent Expressions

Rational exponents have the same properties as integer exponents.

Properties of Rational Exponents

For all nonzero real numbers *a* and *b* and rational numbers *m* and *n*

Words	Numbers	Algebra
Product of Powers Property To multiply powers with the same base, add the exponents.	$12^{\frac{1}{2}} \cdot 12^{\frac{3}{2}} = 12^{\frac{1}{2}+\frac{3}{2}} = 12^2 = 144$	$a^m \cdot a^n = a^{m+n}$
Quotient of Powers Property To divide powers with the same base, subtract the exponents.	$\frac{125^{\frac{2}{3}}}{125^{\frac{1}{3}}} = 125^{\frac{2}{3}-\frac{1}{3}} = 125^{\frac{1}{3}} = 5$	$\frac{a^m}{a^n} = a^{m-n}$
Power of a Power Property To raise one power to another, multiply the exponents.	$\left(8^{\frac{2}{3}}\right)^3 = 8^{\frac{2}{3}\cdot 3} = 8^2 = 64$	$(a^m)^n = a^{m \cdot n}$
Power of a Product Property To find a power of a product, distribute the exponent.	$(16 \cdot 25)^{\frac{1}{2}} = 16^{\frac{1}{2}} \cdot 25^{\frac{1}{2}} = 4 \cdot 5 = 20$	$(ab)^m = a^m b^m$
Power of a Quotient Property To find the power of a qoutient, distribute the exponent.	$\left(\frac{16}{81}\right)^{\frac{1}{4}} = \frac{16^{\frac{1}{4}}}{81^{\frac{1}{4}}} = \frac{2}{3}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Example 1 **Simplify the expression. Assume that all variables are positive. Exponents in simplified form should all be positive.**

(A) **a.** $25^{\frac{3}{5}} \cdot 25^{\frac{7}{5}}$

$= 25^{\frac{3}{5}+\frac{7}{5}}$ Product of Powers Prop.

$= 25^2$ Simplify.

$= 625$

b. $\frac{8^{\frac{1}{3}}}{8^{\frac{2}{3}}}$

$= 8^{\frac{1}{3}-\frac{2}{3}}$ Quotient of Powes Prop.

$= 8^{-\frac{1}{3}}$ Simplify.

$= \frac{1}{8^{\frac{1}{3}}}$ Definition of neg. power

$= \frac{1}{2}$ Simplify.

Explain 1

Simplifying Rational-Exponent Expressions

Avoid Common Errors

Students may, in error, multiply or divide the common bases in a product or quotient of powers. Help them to see that the simplified product (or quotient) represents factors of the common base. Use a numerical example with integer exponents, such as $2^3 \cdot 2^4$, to help students see why this is so.

Questioning Strategies

How do you multiply powers with the same base when the exponents are rational? Add the exponents, and write the result as a power of the common base.

How do you divide powers with the same base when the exponents are rational? Subtract the exponents, and write the result as a power of the common base.

Integrate Technology

Students can use a graphing calculator to check their work. Review the correct use of parentheses when entering expressions containing rational exponents. Also, encourage them to use parentheses around the numerator and the denominator of a quotient of expressions.

Differentiate Instruction

Communicating Math

For many students, the descriptive sentences in the first column of each table of properties will be the most helpful when applying the various properties. Have students read these out loud, and use the examples in the second column to check for understanding. Encourage students to memorize the sentences, rewording them in their own words, as necessary, to clarify the instruction. Then show them how repeating the sentence that applies, at each step of the simplification process, can provide the guidance they need to correctly apply each property.

Explain 2

Simplifying Radical Expressions Using the Properties of Exponents

Questioning Strategies

How does knowing the relationship between roots and expressions containing rational exponents help you to simplify expressions containing radicals? Convert each radical to an expression containing a rational exponent, and then apply the rules for exponents.

How do the properties of exponents help you to multiply two radicals that have the same radicands but different indices? Convert each radical to an expression containing a rational exponent, add the exponents, and then write the result as a power of the common radicand.

Integrate Math Processes and Practices

Focus on Using and Evaluating Logical Reasoning

MPP3 Enhance students' understanding of the properties and relationships used to simplify these expressions by having them explain each step, identifying the properties being applied. You may want to provide students with additional examples, then have them explain the steps of their own, and possibly each other's, work.

B **a.** $\left(\dfrac{y^{\frac{4}{3}}}{16y^{\frac{2}{3}}}\right)^{\frac{3}{2}}$

$= \left(\dfrac{y^{\frac{4}{3}-\frac{2}{3}}}{16}\right)^{\frac{3}{2}}$ Quotient of Powers Prop.

$= \left(\dfrac{\boxed{y^{\frac{2}{3}}}}{16}\right)^{\frac{3}{2}}$ Simplify.

$= \dfrac{\left(y^{\frac{2}{3}}\right)^{\frac{3}{2}}}{16^{\frac{3}{2}}}$ Power of a Quotient Prop.

$= \dfrac{y^{\frac{2}{3}\cdot\frac{3}{2}}}{16^{\frac{3}{2}}}$ Power of a Power Prop.

$= \boxed{\dfrac{y}{64}}$ Simplify.

b. $\left(27x^{\frac{3}{4}}\right)^{\frac{2}{3}}$

$= \boxed{27}^{\frac{2}{3}}\left(\boxed{x^{\frac{3}{4}}}\right)^{\frac{2}{3}}$ Power of a Product Prop.

$= 27^{\frac{2}{3}}\left(x^{\boxed{\frac{3}{4}\cdot\frac{2}{3}}}\right)$ Power of a Power Prop.

$= \boxed{9x^{\frac{1}{2}}}$ Simplify.

Your Turn

Simplify the expression. Assume that all variables are positive. Exponents in simplified form should all be positive.

3. $\left(12^{\frac{2}{3}} \cdot 12^{\frac{4}{3}}\right)^{\frac{3}{2}}$ **1728**

4. $\dfrac{\left(6x^{\frac{1}{3}}\right)^2}{x^{\frac{5}{3}}y}$ $\dfrac{36}{xy}$

Explain 2 Simplifying Radical Expressions Using the Properties of Exponents

When you are working with radical expressions involving nth roots, you can rewrite the expressions using rational exponents and then simplify them using the properties of exponents.

Example 2 **Simplify the expression by writing it using rational exponents and then using the properties of rational exponents. Assume that all variables are positive. Exponents in simplified form should all be positive.**

A $x\left(\sqrt[3]{2y}\right)\left(\sqrt[3]{4x^2y^2}\right)$

$= x(2y)^{\frac{1}{3}}\left(4x^2y^2\right)^{\frac{1}{3}}$ Write using rational exponents.

$= x\left(2y \cdot 4x^2y^2\right)^{\frac{1}{3}}$ Power of a Product Property

$= x\left(8x^2y^3\right)^{\frac{1}{3}}$ Product of Powers Property

$= x\left(2x^{\frac{2}{3}}y\right)$ Power of a Product Property

$= 2x^{\frac{5}{3}}y$ Product of Powers Property

Language Support EL

Communicate Math

Have students work in pairs. The first student explains the steps for simplifying rational exponents to the second student, including the properties involved. The second student takes notes and writes down the steps, and repeats them back using his or her own words. Students switch roles and repeat the procedure for radical expressions involving nth roots.

B $\dfrac{\sqrt{64y}}{\sqrt[3]{64y}}$

$= \dfrac{(64y)^{\frac{1}{2}}}{(64y)^{\frac{1}{3}}}$ Write using rational exponents.

$= (64y)^{\boxed{\frac{1}{2} - \frac{1}{3}}}$ Quotient of Powers Property

$= (64y)^{\boxed{\frac{1}{6}}}$ Simplify.

$= \boxed{64^{\frac{1}{6}}y^{\frac{1}{6}}}$ Power of a Product Property

$= \boxed{2y^{\frac{1}{6}}}$ Simplify.

Your Turn

Simplify the expression by writing it using rational exponents and then using the properties of rational exponents.

5. $\dfrac{\sqrt{x^3}}{\sqrt[3]{x^2}}$ $x^{\frac{5}{6}}$

6. $\sqrt[6]{16^3} \cdot \sqrt[4]{4^6} \cdot \sqrt[3]{8^2}$ **128**

Explain 3 Simplifying Radical Expressions Using the Properties of n^{th} Roots

From working with square roots, you know, for example, that $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$ and $\dfrac{\sqrt{8}}{\sqrt{2}} \cdot = \sqrt{\dfrac{8}{2}} = \sqrt{4} = 2$. The corresponding properties also apply to nth roots.

Properties of *n*th Roots

For a > 0 and b > 0

Words	Numbers	Algebra
Product Property of Roots The *n*th root of a product is equal to the product of the *n*th roots.	$\sqrt[3]{16} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
Quotient Property of Roots The *n*th root of a Quotient is equal to the Quotient of the *n*th roots.	$\sqrt{\dfrac{25}{16}} = \dfrac{\sqrt{25}}{\sqrt{16}} = \dfrac{5}{4}$	$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Explain 3

Simplifying Radical Expressions Using the Properties of *n*th Roots

Questioning Strategies

How do you rationalize a denominator that contains an *n*th root? Multiply the numerator and denominator of the fraction by the *n*th root of enough factors of the radicand to create a perfect *n*th root.

Avoid Common Errors

After learning the product and quotient properties for *n*th roots, students may assume there are similar properties for sums and differences. Show students, by numerical example, that $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ and $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$ for $a, b > 0$.

Integrate Math Processes and Practices
Focus on Abstract and Quantitative Reasoning

MPP2 Have each student work with a partner to use the relationship between radicals and rational exponents to derive the quotient property of roots, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$. Have students justify each step of their derivations.

Example 3 Simplify the expression using the properties of nth roots. Assume that all variables are positive. Rationalize any irrational denominators.

(A) $\sqrt[3]{256x^3y^7}$

$\sqrt[3]{256x^3y^7}$

$= \sqrt[3]{2^8 \cdot x^3y^7}$ — Write 256 as a power.

$= \sqrt[3]{2^6 \cdot x^3y^6} \cdot \sqrt[3]{2^2 \cdot y}$ — Product Property of Roots

$= \sqrt[3]{2^6} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{y^6} \cdot \sqrt[3]{4y}$ — Factor out perfect cubes.

$= 4xy^2\sqrt[3]{4y}$ — Simplify.

(B) $\sqrt[4]{\frac{81}{x}}$

$\sqrt[4]{\frac{81}{x}}$

$= \frac{\sqrt[4]{81}}{\sqrt[4]{x}}$ — $\boxed{\text{Quotient Property of Roots}}$

$= \frac{\boxed{3}}{\sqrt[4]{x}}$ — Simplify.

$= \frac{3}{\sqrt[4]{x}} \cdot \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}}$ — Rationalize the denominator.

$= \frac{3\sqrt[4]{x^3}}{\sqrt[4]{x^4}}$ — $\boxed{\text{Product Property of Roots}}$

$= \boxed{\frac{3\sqrt[4]{x^3}}{x}}$ — Simplify.

Reflect

7. In Part B, why was $\sqrt[4]{x^3}$ used when rationalizing the denominator? What factor would you use to rationalize a denominator of $\sqrt[5]{4y^3}$? **It was chosen to make the product of the radicands be a perfect fourth power so that the fourth root could be taken; $\sqrt[5]{8y^2}$.**

Your Turn

Simplify the expression using the properties of nth roots. Assume that all variables are positive.

8. $\sqrt[3]{216x^{12}y^{15}}$ **$6x^4y^5$**

9. $\sqrt[4]{\frac{16}{x^{14}}}$ **$\frac{2\sqrt[4]{x^2}}{x^4}$**

Explain 4 Rewriting a Radical-Function Model

When you find or apply a function model involving rational powers or radicals, you can use the properties in this lesson to help you find a simpler expression for the model.

Ⓐ **Manufacturing** A can that is twice as tall as its radius has the minimum surface area for the volume it contains. The formula $S = 6\pi\left(\frac{V}{2\pi}\right)^{\frac{2}{3}}$ expresses the surface area of a can with this shape in terms of its volume.

a. Use the properties of rational exponents to simplify the expression for the surface area. Then write the approximate model with the coefficient rounded to the nearest hundredth.

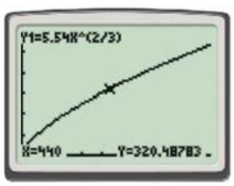
b. Graph the model using a graphing calculator. What is the surface area in square centimeters for a can with a volume of 440 cm³?

a.

	$S = 6\pi\left(\frac{V}{2\pi}\right)^{\frac{2}{3}}$
Power of a Quotient Property	$= 6\pi \cdot \frac{V^{\frac{2}{3}}}{(2\pi)^{\frac{2}{3}}}$
Group Powers of 2π.	$= \frac{3(2\pi)}{(2\pi)^{\frac{2}{3}}} \cdot V^{\frac{2}{3}}$
Quotient of Powers Property	$= 3(2\pi)^{1-\frac{2}{3}} \cdot V^{\frac{2}{3}}$
Simplify.	$= 3(2\pi)^{\frac{1}{3}} \cdot V^{\frac{2}{3}}$
Use a calculator.	$\approx 5.54V^{\frac{2}{3}}$

A simplified model is $S = 3(2\pi)^{\frac{1}{3}} \cdot V^{\frac{2}{3}}$, which gives $S \approx 5.54V^{\frac{2}{3}}$.

b.



The surface area is about 320 cm².

Explain 4

Rewriting a Radical-Function Model

Questioning Strategies

How can you check that the simplified form of the expression is equivalent to the original expression? You could graph both expressions as functions on a graphing calculator and make sure their graphs are the same. You could also evaluate both expressions for several values and make sure the resulting values are the same.

Elaborate

Questioning Strategies

Can you use the properties of rational exponents to simplify $\sqrt{a} \cdot \sqrt[3]{b}$? Explain. No; in rational exponent form, the expression is $a^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$. Because the bases are different, the product of powers property does not apply, and the expression cannot be simplified.

Summarize The Lesson

How can the properties of exponents be applied to the simplification of expressions containing rational exponents and to those containing radicals? For expressions containing rational exponents, the properties of exponents can be applied directly. For radical expressions, convert the radical expressions to exponent form using the fact that $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, and then apply the properties.

Answers

12. You can multiply the expression by a form of 1 so that the denominator of the resulting expression has an exponent that is an integer. For example, for $\frac{6}{x^{\frac{2}{5}}}$, multiply the numerator and denominator by $x^{\frac{3}{5}}$: $\frac{6}{x^{\frac{2}{5}}} \cdot \frac{x^{\frac{3}{5}}}{x^{\frac{3}{5}}} = \frac{6x^{\frac{3}{5}}}{x^{\frac{2}{5}+\frac{3}{5}}} = \frac{6x^{\frac{3}{5}}}{x}$.

13. By definition, the nth root of a number b is the number whose nth power is b. So the nth root of a^n is the number whose nth power is a^n, or a. Using rational exponents, the nth root is indicated by the exponent $\frac{1}{n}$, so $\sqrt[n]{a^n} = \left(a^n\right)^{\frac{1}{n}} = a^{n \cdot \frac{1}{n}} = a^{\frac{n}{n}} = a^1 = a$.

14. The nth root is indicated by the exponent $\frac{1}{n}$, so $\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

B **Commercial fishing** The buoyancy of a fishing float in water depends on the volume of air it contains. The radius of a spherical float as a function of its volume is given by $r = \sqrt[3]{\frac{3V}{4\pi}}$.

a. Use the properties of roots to rewrite the expression for the radius as the product of a coefficient term and a variable term. Then write the approximate formula with the coefficient rounded to the nearest hundredth.

b. What should the radius be for a float that needs to contain 4.4 ft^3 of air to have the proper buoyancy?

a. $r = \sqrt[3]{\frac{3V}{4\pi}}$

Rewrite radicand. $= \sqrt[3]{\frac{3}{4\pi} \cdot \boxed{V}}$

Product Property of Roots $= \sqrt[3]{\frac{3}{4\pi}} \cdot \boxed{\sqrt[3]{V}}$

Use a calculator $\approx \boxed{0.62\sqrt[3]{V}}$

The rewritten formula is $r = \boxed{\sqrt[3]{\frac{3}{4\pi}} \cdot \sqrt[3]{V}}$, which gives $r \approx \boxed{0.62\sqrt[3]{V}}$.

b. Substitute 4.4 for V.

$r = 0.62\sqrt[3]{4.4} \approx \boxed{1.02}$

The radius is about 1.0 feet.

Reflect

10. **Discussion** What are some reasons you might want to rewrite an expression involving radicals into an expression involving rational exponents?

10. By rewriting radical expressions, especially complicated ones and those involving *n*th roots and powers in the radicand, rewriting using rational exponents lets you use the properties of rational exponents to make simplification easier. Also, rational exponents make it easier to enter an expression into a calculator for evaluation or graphing.

Your Turn

11. The surface area as a function of volume for a box with a square base and a height that is twice the side length of the base is $S = 10\left(\frac{V}{2}\right)^{\frac{2}{3}}$. Use the properties of rational exponents to simplify the expression for the surface area so that no fractions are involved. Then write the approximate model with the coefficient rounded to the nearest hundredth. **The expression is $S = 5 \cdot 2^{\frac{1}{3}}V^{\frac{2}{3}}$, which gives $S = 6.30V^{\frac{2}{3}}$.**

Elaborate

12–15. See margin.

12. In problems with a radical in the denominator, you rationalized the denominator to remove the radical. What can you do to remove a rational exponent from the denominator? Explain by giving an example.

13. Show why $\sqrt[n]{a^n}$ is equal to a for all natural numbers a and n using the definition of nth roots and using rational exponents.

14. Show that the Product Property of Roots is true using rational exponents.

15. **Essential Question Check-In** Describe the difference between applying the Power of a Power Property and applying the Power of a Product Property for rational exponents using an example that involves both properties.

---

15. Possible answer: Consider the expression $\left(4^{\frac{2}{3}}x^{\frac{4}{3}}\right)^{\frac{3}{4}}$. This is the $\frac{3}{4}$ power of the product of $4^{\frac{2}{3}}$ and $x^{\frac{4}{3}}$. The result is the product of the $\frac{3}{4}$ power of each factor: $\left(4^{\frac{2}{3}}\right)^{\frac{3}{4}}\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}}$. This expression contains the $\frac{3}{4}$ power of the power $4^{\frac{2}{3}}$ and the $\frac{3}{4}$ power of the power $x^{\frac{4}{3}}$. Simplifying then gives $\left(4^{\frac{2}{3}}\right)^{\frac{3}{4}}\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} = 4^{\frac{1}{2}} \cdot x^1 = 2x$

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

Simplify the expression. Assume that all variables are positive. Exponents in simplified form should all be positive.

1. $\left(\left(\frac{1}{16}\right)^{-\frac{2}{3}}\right)^{\frac{3}{4}}$ **4**

2. $\frac{x^{\frac{1}{3}} \cdot x^{\frac{5}{6}}}{x^{\frac{1}{6}}}$ **x**

3. $\frac{9^{\frac{3}{2}} \cdot 9^{\frac{1}{2}}}{9^{-2}}$ **6561**

4. $\left(\frac{16^{\frac{5}{3}}}{16^{\frac{5}{6}}}\right)^{\frac{9}{5}}$ **64**

5. $\frac{2xy}{\left(x^{\frac{1}{3}}y^{\frac{2}{3}}\right)^{\frac{3}{2}}}$ **$2x^{\frac{1}{2}}$**

6. $\frac{3y^{\frac{3}{4}}}{2xy^{\frac{3}{2}}}$ **$\frac{3y^{\frac{1}{4}}}{2xy}$**

Simplify the expression by writing it using rational exponents and then using the properties of rational exponents. Assume that all variables are positive. Exponents in simplified form should all be positive.

7. $\sqrt[4]{25} \cdot \sqrt[3]{5}$ **$5^{\frac{5}{6}}$**

8. $\frac{\sqrt[4]{2^{-2}}}{\sqrt[6]{2^{-9}}}$ **2**

9. $\frac{\sqrt[4]{3^3} \cdot \sqrt[3]{x^2}}{\sqrt{3x}}$ **$3^{\frac{1}{4}}x^{\frac{1}{6}}$**

10. $\frac{\sqrt[4]{x^4y^6} \cdot \sqrt{x^6}}{y}$ **$x^4y^{\frac{1}{2}}$**

11. $\frac{\sqrt[6]{s^4t^9}}{\sqrt[3]{st}}$ **$s^{\frac{1}{3}}t^{\frac{7}{6}}$**

12. $\sqrt[4]{27} \cdot \sqrt{3} \cdot \sqrt[6]{81^3}$ **$27 \cdot 3^{\frac{1}{4}}$**

Simplify the expression using the properties of nth roots. Assume that all variables are positive. Rationalize any irrational denominators.

13. $\frac{\sqrt[4]{36} \cdot \sqrt[4]{216}}{\sqrt[4]{6}}$ **6**

14. $\sqrt[4]{4096x^6y^8}$ **$8xy^2\sqrt{x}$**

15. $\frac{\sqrt[3]{x^8y^4}}{\sqrt[3]{x^2y}}$ **x^2y**

16. $\sqrt[5]{\frac{125}{w^6}} \cdot \sqrt[5]{25v}$ **$\frac{5\sqrt[5]{vw^4}}{w^2}$**

Exercise	Depth of Knowledge (D.O.K.)	Math Processes and Practices
1	**1** Recall of Information	**MPP3** Using and Evaluating Logical Reasoning
2–16	**1** Recall of Information	**MPP5** Using Mathematical Tools
17–20	**2** Skills/Concepts	**MPP4** Mathematical Modeling
21	**2** Skills/Concepts H.O.T.	**MPP3** Using and Evaluating Logical Reasoning
22–23	**3** Strategic Thinking H.O.T.	**MPP3** Using and Evaluating Logical Reasoning

Evaluate

Personal Math Trainer

Assignment Guide

Level	Concepts and Skills	Practice
Basic	Explore	N/A
	Example 1	Exercises 1–6
	Example 2	Exercises 7–12
	Example 3	Exercises 14–15
	Example 4	Exercise 18
	H.O.T.	Exercise 21
Average	Explore	N/A
	Example 1	Exercises 1–6
	Example 2	Exercises 7–12
	Example 3	Exercises 13–16
	Example 4	Exercises 17–18
	H.O.T.	Exercises 21–22
Advanced	Explore	N/A
	Example 1	Exercises 2, 4, 6
	Example 2	Exercises 8, 10, 12
	Example 3	Exercises 13–16
	Example 4	Exercises 17–18, 20
	H.O.T.	Exercises 21–23

Real World Problems

Questioning Strategies

When do you add the exponents on two expressions? when the expressions have the same base and they are being multiplied

When do you apply the power of a power property? when an expression containing an exponent is being raised to another exponent

Integrate Math Processes and Practices

Focus on Abstract and Quantitative Reasoning

MPP2 Discuss with students that there is often more than one way to go about simplifying these types of expressions. Encourage students to be aware of this, and to use one method to check their results found using a different method.

Visual Cues

For expressions that involve applying the power of a product property or the power of a quotient property, suggest that students use arrows to show the exponent being applied to each factor in the product or to each factor in the numerator and denominator in the quotient. In this way, students may avoid errors such as forgetting to apply the exponent to a numerical coefficient or to a variable that does not contain an exponent.

Avoid Common Errors

Students may make errors in rationalizing denominators when the index is greater than 2. For example, when trying to rationalize an expression such as $\frac{\sqrt[5]{2}}{\sqrt[5]{x}}$, they may multiply by $\frac{\sqrt[5]{x}}{\sqrt[5]{x}}$, instead of $\frac{\sqrt[5]{x^4}}{\sqrt[5]{x^4}}$, as if the index were 2 instead of 5. Help them to see that this choice does not make the radicand a perfect fifth, which is the goal of rationalizing this denominator. Reinforce that the resulting exponent must be a multiple of the index.

17. **Weather** The volume of a sphere as a function of its surface area is given by $V = \frac{4\pi}{3}\left(\frac{S}{4\pi}\right)^{\frac{3}{2}}$.

a. Use the properties of roots to rewrite the expression for the volume as the product of a simplified coefficient term (with positive exponents) and a variable term. Then write the approximate formula with the coefficient rounded to the nearest thousandth.

b. A spherical weather balloon has a surface area of 500 ft². What is the approximate volume of the balloon?

a. $V = \frac{1}{6\pi^{\frac{1}{2}}} \cdot S^{\frac{3}{2}}$

$\approx 0.094 S^{\frac{3}{2}}$

b. 1050 ft^3

18. **Amusement parks** An amusement park has a ride with a free fall of 128 feet. The formula $t = \sqrt{\frac{2d}{g}}$ gives the time t in seconds it takes the ride to fall a distance of d feet. The formula $v = \sqrt{2gd}$ gives the velocity v in feet per second after the ride has fallen d feet. The letter g represents the gravitational constant. **See below.**

a. Rewrite each formula so that the variable d is isolated. Then simplify each formula using the fact that $g \approx 32$ ft/s².

b. Find the time it takes the ride to fall halfway and its velocity at that time. Then find the time and velocity for the full drop.

c. What is the ratio of the time it takes for the whole drop to the time it takes for the first half? What is the ratio of the velocity after the second half of the drop to the velocity after the first half? What do you notice?

19. Which choice(s) is/are equivalent to $\sqrt{2}$? **A, C, D, and E**

A. $\left(\sqrt[8]{2}\right)^4$

B. $\frac{2^3}{2^{-\frac{5}{2}}}$

C. $\left(4^{\frac{2}{3}} \cdot 2^{\frac{2}{3}}\right)^{\frac{1}{4}}$

D. $\frac{\sqrt[3]{2^2}}{\sqrt[6]{2}}$

E. $\frac{\sqrt{2^{-\frac{3}{4}}}}{\sqrt{2^{-\frac{7}{4}}}}$

18. a. $t = \sqrt{\frac{2}{g}} \cdot \sqrt{d}$, or $t \approx \frac{1}{4}\sqrt{d}$, and $v = \sqrt{2g} \cdot \sqrt{d}$, or $v \approx 8\sqrt{d}$.

b. halfway: $t = 2$ s and $v = 64$ ft/s; full drop: $t = 2\sqrt{2}$ s, or about 2.8 s, and $v = 64\sqrt{2}$ ft/s, or about 91 ft/s

c. $\frac{\text{whole time}}{\text{1st half time}} = \sqrt{2} \approx 1.41$; $\frac{\text{velocity at end}}{\text{velocity halfway}} = \sqrt{2} \approx 1.41$; the ratios are the same

20. Home Heating A propane storage tank for a home is shaped like a cylinder with hemispherical ends, and a cylindrical portion length that is 4 times the radius. The formula $S = 12\pi\left(\frac{3V}{16\pi}\right)^{\frac{2}{3}}$ expresses the surface area of a tank with this shape in terms of its volume.

a. Use the properties of rational exponents to rewrite the expression for the surface area so that the variable V is isolated. Then write the approximate model with the coefficient rounded to the nearest hundredth. **See below.**

b. Graph the model using a graphing calculator. What is the surface area in square feet for a tank with a volume of 150 ft^3 ? **The surface area is about 163 ft^2.**

H.O.T. Focus on Higher Order Thinking

21. Critique Reasoning Aaron's work in simplifying an expression is shown. What mistake(s) did Aaron make? Show the correct simplification.

$$625^{-\frac{1}{3}} \div 625^{-\frac{4}{3}}$$
$$= 625^{-\frac{1}{3} - \left(-\frac{4}{3}\right)}$$
$$= 625^{-\frac{1}{3}\left(-\frac{3}{4}\right)}$$
$$= 625^{\frac{1}{4}}$$
$$= 5$$

Aaron incorrectly applied the Quotient of Powers Property. He should have subtracted the exponents.

$$\mathbf{625^{-\frac{1}{3}} \div 625^{-\frac{4}{3}}}$$
$$\mathbf{= 625^{-\frac{1}{3} - \left(-\frac{4}{3}\right)} = 625^{-\frac{1}{3} + \frac{4}{3}} = 625^1 = 625}$$

22. Critical Thinking Use the definition of nth root to show that the Product Property of Roots is true, that is, that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$. (Hint: Begin by letting x be the nth root of a and letting y be the nth root of b.) **See below.**

23. Critical Thinking For what real values of a is $\sqrt[4]{a}$ greater than a? For what real values of a is $\sqrt[5]{a}$ greater than a?
for $0 < a < 1$; for $a < -1$ or $0 < a < 1$

20a. $S = 12\pi\left(\frac{3}{16\pi}\right)^{\frac{2}{3}} \cdot V^{\frac{2}{3}} \approx 5.76V^{\frac{2}{3}}$

22. **Let x be the nth root of a and let y be the nth root of b. Then by the definition of nth root, $a = x^n$ and $b = y^n$. So, $ab = x^ny^n = (xy)^n$. This means by definition that xy is the nth root of ab, or $\sqrt[n]{ab} = xy$. But $xy = \sqrt[n]{a} \cdot \sqrt[n]{b}$, so $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.**

Technology

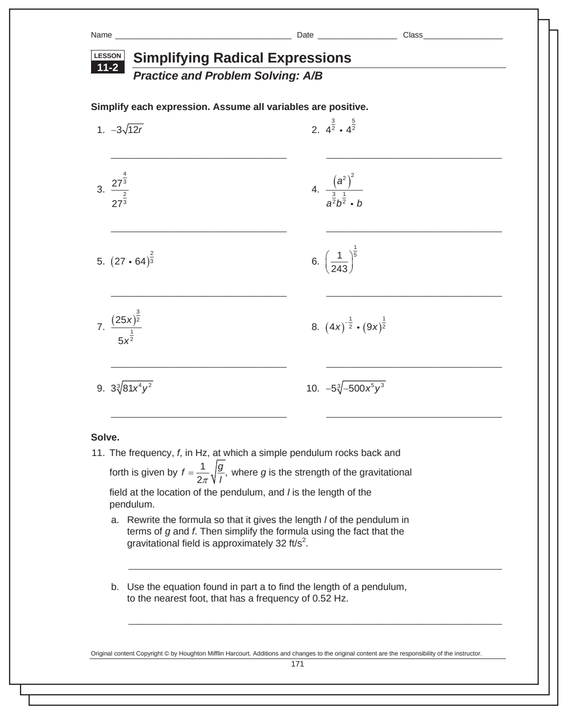
Encourage students to use a graphing calculator to check their work. To check problems involving the simplification of variable expressions, suggest that students assign values to the variables and use the calculator to check that the value of the original expression and the value of the simplified expression are the same when evaluated for the same values of the variables.

Journal

Have students describe how the properties of nth roots are similar to the corresponding properties of rational exponents.

Online Resources

- Practice and Problem Solving (three forms)
- Reteach
- Reading Strategies
- Success for English Learners



Name ______ Date ______ Class ______

LESSON 11-2 **Simplifying Radical Expressions**
Practice and Problem Solving: A/B

Simplify each expression. Assume all variables are positive.

1. $-3\sqrt{12r}$
2. $4^{\frac{3}{2}} \cdot 4^{\frac{5}{2}}$
3. $\frac{27^{\frac{4}{3}}}{27^{\frac{2}{3}}}$
4. $\frac{(a^2)^2}{a^{\frac{3}{2}}b^{\frac{1}{2}} \cdot b}$
5. $(27 \cdot 64)^{\frac{2}{3}}$
6. $\left(\frac{1}{243}\right)^{\frac{1}{5}}$
7. $\frac{(25x)^{\frac{3}{2}}}{5x^{\frac{1}{2}}}$
8. $(4x)^{-\frac{1}{2}} \cdot (9x)^{\frac{1}{2}}$
9. $3\sqrt[3]{81x^4y^2}$
10. $-5\sqrt[3]{-500x^5y^3}$

Solve.

11. The frequency, f, in Hz, at which a simple pendulum rocks back and forth is given by $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$, where g is the strength of the gravitational field at the location of the pendulum, and l is the length of the pendulum.
 a. Rewrite the formula so that it gives the length l of the pendulum in terms of g and f. Then simplify the formula using the fact that the gravitational field is approximately 32 ft/s^2.
 b. Use the equation found in part a to find the length of a pendulum, to the nearest foot, that has a frequency of 0.52 Hz.

171

Avoid Common Errors

When dividing one term by another, some students may divide the exponents instead of subtracting. Explain that to simplify $\frac{4\pi}{(4\pi)^{\frac{2}{3}}}$, you subtract the exponents: $1 - \frac{2}{3} = \frac{1}{3}$. The simplified term would then be $(4\pi)^{\frac{1}{3}}$.

Integrate Math Processes and Practices

Focus on Mathematical Modeling

MPP4 Have students discuss why the final exponent of V turns out to be $\frac{2}{3}$. Have them relate the numbers in the rational exponent to the geometry of the situation. Have them discuss whether the exponent would be $\frac{2}{3}$ if the shape were something other than a sphere.

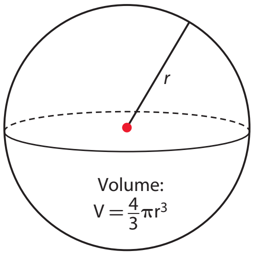
Lesson Performance Task Scoring Rubric

Points	Criteria
2	Student correctly solves the problem and explains his/her reasoning.
1	Student shows good understanding of the problem but does not fully solve or explain his/her reasoning.
0	Student does not demonstrate understanding of the problem.

Lesson Performance Task

You've been asked to help decorate for a school dance, and the theme chosen is "The Solar System." The plan is to have a bunch of papier-mâché spheres serve as models of the planets, and your job is to paint them. All you're told are the volumes of the individual spheres, but you need to know their surface areas so you can get be sure to get enough paint. How can you write a simplified equation using rational exponents for the surface area of a sphere in terms of its volume?

(The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the formula for the surface area of a sphere is $A = 4\pi r^2$.)



Solve the volume formula for *r*.

$$\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = r$$

Substitute the value for *r* into the formula for surface area and simplify.

The surface area in terms of the volume is equal to $A = (4\pi)^{\frac{1}{3}}(3V)^{\frac{2}{3}}$.

Extension Activity

Have students research the equations for the surface area and volume for a truncated *icosidodecahedron*, a 32-face solid similar to a soccer ball. Have students derive an equation for the surface area in terms of the volume, and then have them discuss whether they will need more or less paint to cover this shape, compared to painting a sphere.